Unfolding mixed-symmetry fields in AdS and the BMV conjecture: II. Oscillator realization

Nicolas Boulanger,[1] **Carlo Iazeolla**[2] **and Per Sundell**

Scuola Normale Superiore,
Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail: nicolas.boulanger@sns.it, carlo.iazeolla@sns.it,
p.sundell@sns.it

ABSTRACT: Following the general formalism presented in arXiv:0812.3615 — referred to as Paper I — we derive the unfolded equations of motion for tensor fields of arbitrary shape and mass in constantly curved backgrounds by radial reduction of Skvortsov's equations in one higher dimension. The complete unfolded system is embedded into a single master field, valued in a tensorial Schur module realized equivalently via either bosonic (symmetric basis) or fermionic (anti-symmetric basis) vector oscillators. At critical masses the reduced Weyl zero-form modules become indecomposable. We explicitly project the latter onto the submodules carrying Metsaev's massless representations. The remainder of the reduced system contains a set of Stückelberg fields and dynamical potentials that leads to a smooth flat limit in accordance with the Brink-Metsaev-Vasiliev (BMV) conjecture. In the unitary massless cases in AdS, we identify the Alkalaev-Shaynkman-Vasiliev frame-like potentials and explicitly disentangle their unfolded field equations.

Contents

1 Introduction

In a companion paper [1] referred to as Paper I, we introduced some general tools and notation adapted to the unfolded description of tensor fields propagating in constantly curved backgrounds. Such an analysis was initiated by Alkalaev, Shaynkman and Vasiliev (ASV), who proposed an action in frame-like formalism for mixed-symmetry fields in $(A)dS_D$ spacetimes [2], and was recently performed in the case of Minkowski spacetime by Skvortsov [3], who provided the corresponding unfolded field equations. In the present Paper II, we use these tools together with an oscillator formulation of Schur modules in order to effectively write down unfolded field equations for arbitrary tensor fields freely propagating in AdS_D spacetime. Metric-like and partially gauge-fixed equations have previously been given by Metsaev in [4, 5]. For some recent works on mixed-symmetry fields in AdS, see [6–9] and references therein.

As in Paper I, we use the unfolded formalism [10–12] whereby the concepts of spacetime, dynamics and observables are derived from free differential algebras [13–16]. The key features are that (i) equations of motion, Bianchi identities as well as definitions of auxiliary fields are encoded into flatness conditions on complete sets of generalized curvatures, including in general an infinite set of zero-forms called *Weyl zero-forms*; (ii) the diffeomorphism invariance is manifest (this symmetry is then broken spontaneously by given solutions); and (iii) the gauge invariance is ensured by consistency conditions on the coupling constants in the curvatures that can be solved using algebraic techniques for deformations of associative algebras (including Lie algebras) and their representations. This powerful framework is instrumental in controlling the field content and symmetries of higher-spin gauge theory and underlies Vasiliev's fully nonlinear field equations for totally symmetric gauge fields [17, 18]. It is therefore likely to be helpful also in addressing the challenging issue of interacting mixed-symmetry gauge fields.

Tensor fields of mixed symmetry exhibit, already at the free level, peculiarities that are absent in the "rectangular" case, including symmetric tensor fields and ordinary p-forms.

Such fields must be considered in flat spacetime as soon as $D \geqslant 6$ and in constantly curved spacetime as soon as $D \geqslant 4$ (in accordance with the analysis done in [19], unitary massless mixed-symmetry two-row tensor fields in AdS_4 decompose in the flat limit into topological dittos plus one massless field in $\mathbb{R}^{1,3}$).

As far as free tensor gauge fields in flat spacetime of dimension $D \geqslant 4$ are concerned, a Lagrangian formulation was proposed some time ago by Labastida [20]. That the corresponding equations of motion indeed propagate the proper massless degrees of freedom was understood later [21] — see [22] for a review and references. The proof of the propagation of the proper massless physical degrees of freedom crucially relies on the properties of the generalized curvature K and its traceless part, defined in [23]. The local wave equation proposed in [23, 24] for an arbitrary tensor gauge field in flat spacetime can be seen as the generalization to arbitrary dimensions of the Bargmann-Wigner equation [25] proposed in $D = 4$, and were therefore called "generalized Bargmann-Wigner equations" in [24] and henceforth. Let us finally mention that a trace-unconstrained version of Labastida's formulation has appeared in [26], though we shall not make direct contact with this off-shell formulation here.

In Paper I, we reviewed and extended the generalized Bargmann-Wigner equations to constantly curved spacetimes, translating them into the unfolding language which facilitates their integration whereupon p-form variables arise that generalize the vielbein and Lorentz-connection of spin-2 theory. The results, that complete the analysis of the pioneering work [2], are given here, comprising the complete infinite-dimensional Weyl zero-form module as well as the finite-dimensional p-form modules.

As we mentioned previously, the unfolded presentation of Labastida's formalism was given recently by Skvortsov [3, 27] and results in a system consisting of p-forms ($p \geqslant 0$) that are traceless Lorentz tensors of various symmetry types determined by the Young diagram of the massless metric-like field. The p-forms with fixed p constitute on-shell $\mathfrak{iso}(1, D-1)$-modules that are finite-dimensional for $p > 0$ and infinite-dimensional for $p = 0$ — the aforementioned Weyl zero-form module. The first-order action [27] directly generalizes Vasiliev's first-order action [28] for Fronsdal fields in flat space [29] to arbitrarily-shaped gauge fields. In the present paper we review and reformulate Skvortsov's unfolded equations in terms of master fields taking their values in generalized Schur modules realized explicitly using oscillators and Fock spaces. We use this reformulation in order to extend Skvortsov's formulation to AdS_D, thereby making contact with the equations and the p-form module proposed by ASV in [2, 30, 31], see also [32].

The present analysis in AdS_D allows us to unfold a conjecture due to Brink, Metsaev and Vasiliev. The BMV conjecture [19] anticipates a field-theoretic realization of an AdS mixed-symmetry gauge field with shape Θ, $\varphi(\Lambda; \Theta)$, in terms of an "unbroken" gauge field plus a set of Stückelberg fields $\{\chi(\Lambda; \Theta')\}$ that break the gauge symmetries associated with all blocks but one, in such a way that the combined system has a smooth flat limit — in the sense that the number of local degrees of freedom is conserved — given by the direct sum $\varphi(\Lambda{=}0; \Theta) \oplus \bigoplus_{\Theta'} \chi(\Lambda{=}0; \Theta')$ of irreducible gauge fields in $\mathbb{R}^{1,D-1}$. More precisely, the set $\{\Theta'\}$ should be given by the reduction of the $\mathfrak{so}(D-1)$-tensor of shape Θ under $\mathfrak{so}(D-2)$ subject to the condition that one block, the one associated with the leftover gauge invariance, must remain untouched. In the unitary case, that block must be the uppermost one.

The partially massive nature of mixed-symmetry gauge fields in AdS_D [4, 5] and the dimensional reduction leading to $\{\Theta'\}$ suggest that the Stückelberg fields can be incorporated explicitly via a suitable radial reduction of an unbroken gauge field in $(D+1)$-dimensional flat ambient space with signature $(2, D-1)$. In this paper, we carry out this procedure using the unfolded language, which is readily adapted to dimensional reductions as "world" and "fiber" indices are treated separately from the outset. We stress that our treatment accommodates any combinations of ambient and tangent space signatures, and that the radial reduction allows for arbitrary values of the mass parameter, introduced by constraining the radial derivatives of all p-forms in the unfolded system (see eq. (3.10)). In particular, the reduction allows for general "critical masses" (see items (i)-(iv) in section I.4.3.4), though we shall focus mainly on the case of Metsaev's massless fields in AdS_D, leaving a number of details in other special cases for future work.

The paper is organized as follows: The general formalism underlying the analysis in this paper is contained in Paper I. (We recall some of our notation in appendix A.) In section 2 we review Skvortsov's unfolded equations in $\mathbb{R}^{1,D-1}$ and then cast them into a master-field form suitable for radial reduction using oscillator realizations of Young diagrams. Finally, in section 3 we derive the unfolded equations for general tensor fields in AdS_D, analyze critical limits for the mass parameter and show the resulting smoothness of the flat limit in accordance with the BMV conjecture [19]. In particular, see equations (3.27) and (3.28) for the zero-forms. The appropriate projection to Metsaev's critical cases is given in eq. (3.72), and the corresponding value of the mass parameter in eq. (3.74). Finally, the unfolded equations for the unitary ASV potential are (3.100). Our conclusions and an outlook are presented in section 4. Appendix B contains a review of Howe duality in the context of classical Lie algebras. Appendix C details the radial reduction of the background fields in $\mathbb{R}^{2,D-1}$. Appendix D lists shapes occurring in the computation of the σ^--cohomology groups for ASV potentials with $h_1 = 1$. (We note that the general sigma-minus construction was introduced in [33].) Appendix E shows that some AdS_D-massless lowest-weight unitary representations may arise in tensor products of P bosonic singletons only if $P = 2$. Besides, the Metsaev's mixed-symmetry that may appear have at most six blocks, the first of height one, and are therefore associated with a one-form ASV potential.

2 Tensor gauge fields in flat spacetime

In this section we first review Skvortsov's unfolded formalism for free tensor gauge fields in flat spacetime [3, 27]. We then cast them into a compact master-field form using an oscillator realization of Young tableaux.

2.1 Skvortsov's unfolded equations

The unfolding in D-dimensional Minkowski spacetime of an on-shell tensor gauge field $\varphi(\Theta)$ sitting in the $\mathfrak{m}$-type[1]

$$\Theta = \Big([s_0; h_0], [s_1; h_1], \ldots, [s_B; h_B], [s_{B+1}; h_{B+1}] \Big), \tag{2.1}$$

---

[1]In the following, we shall frequently suppress the labels s_0, s_{B+1}, h_0 and h_{B+1}, in the presentation of Young diagrams associated to dynamical fields.

$$s_0 := \infty > s_1 > \cdots > s_B > s_{B+1} := 0, \tag{2.2}$$

$$h_0 := 0, \quad h_1 \geqslant 1, \quad h_2 \geqslant 1, \ldots, h_{B+1} := \infty \tag{2.3}$$

results in a triangular $\mathfrak{g}_0$-module $\mathfrak{T}(\Theta) = \bigoplus_{q \in \mathbb{Z}} \mathfrak{R}_q(\Theta)$ with indecomposable structure

$$\mathfrak{R}_q|_{\mathfrak{g}_0} = \mathfrak{R}_q^{\mathbf{p}_B+\mathbf{q}} \uplus \mathfrak{R}_q^{\mathbf{p}_{B-1}+\mathbf{q}} \uplus \cdots \uplus \mathfrak{R}_q^{\mathbf{h}_1+\mathbf{q}} \uplus \mathfrak{R}_q^{\mathbf{q}}, \tag{2.4}$$

where $p_I = \sum_{J=1}^{I} h_J$, $(I = 1, \ldots, B)$, $p_0 := 0$. The submodules are given by

$$\mathfrak{R}_q^{\mathbf{p}_I+\mathbf{q}} = \Omega^{\mathbf{p}_I+\mathbf{q}}(U) \otimes \mathcal{T}_{(p_I+1)}^-(\Theta_{[p_I]}^-), \tag{2.5}$$

$$\Theta_{[p_I]}^- = \Big(\underbrace{[s_1-1;h_1], \ldots, [s_I-1;h_I]}_{\text{cut one column}}, \underbrace{[s_{I+1};h_{I+1}+1]}_{\text{add one row}}, [s_{I+2};h_{I+2}], \ldots, [s_B;h_B] \Big), \tag{2.6}$$

which vanishes trivially if $p_I + q < 0$. For $I > 0$ the submodules are finite-dimensional and one has

$$I \geqslant 1 : \mathcal{T}_{(p_I+1)}^-(\Theta_{[p_I]}^-) \cong \mathcal{T}_{(p_I+1)}^+(\Theta_{[p_I]}^+), \tag{2.7}$$

$$\Theta_{[p_I]}^+ = \Big(\underbrace{[s_1-1;h_1], \ldots, [s_{I-1}-1;h_{I-1}], \underbrace{[s_I-1;h_I+1]}_{\text{add one row}}}_{\text{cut one column}}, [s_{I+1};h_{I+1}], \ldots, [s_B;h_B] \Big). \tag{2.8}$$

For $I = 0$ the submodule is infinite-dimensional and defines the twisted-adjoint $\mathfrak{g}_0$-module

$$I = 0 : \mathfrak{R}_q^{\mathbf{q}} := \Omega^{\mathbf{q}}(U) \otimes \mathcal{T}(\Lambda=0; M^2=0; \overline{\Theta}), \quad \mathcal{T}(\Lambda=0; M^2=0; \overline{\Theta}) := \mathcal{T}_{(1)}^-(\Theta_{[p_0]}^-), \tag{2.9}$$

$$\overline{\Theta} = \Theta_{[p_0]}^- = \Big([s_1;h_1+1], [s_2;h_2], \ldots, [s_B;h_B] \Big). \tag{2.10}$$

Upon defining

$$s_{B+1} := 0, \quad s_0 := \infty, \quad s_{I,J} := s_I - s_J, \quad \alpha := k_I + s_{I+1,1}, \quad k_I \in \{0, \ldots, s_{I,I+1}-1\} \tag{2.11}$$

one has

$$\mathfrak{R}_q|_{\mathfrak{m}} = \bigoplus_{\alpha=-s_1}^{\infty} \Omega^{[\mathbf{p}_\alpha+\mathbf{q}]}(U) \otimes \Theta_{[p_\alpha];\alpha}, \tag{2.12}$$

$$\Theta_{[p_I];\alpha} = \Big(\underbrace{[s_1-1;h_1], \ldots, [s_I-1;h_I]}_{\text{cut one column}}, \underbrace{[s_{I+1}+k_I;1]}_{\text{insert one row}}, [s_{I+1};h_{I+1}], \ldots, [s_B;h_B] \Big), \tag{2.13}$$

that is, for fixed $I \in \{0, \ldots, B\}$, the set $\{\Theta_{[p_I];\alpha}\}$ is obtained from Θ by first deleting one column from each of the first I blocks of Θ and then inserting one extra row of variable length between the Ith and $(I+1)$th blocks in compliance with row order (with $s_{B+1} := 0$ and $s_0 := \infty$). In particular, the form of highest degree $\mathbf{p}_B$ sits in the smallest Lorentz type $\widetilde{\Theta} := \Theta_{[\mathbf{p}_B];-s_1}$ given by Θ minus its first column, and the smallest zero-form is the primary Weyl tensor sitting in $\overline{\Theta} =: \Theta_{[0];0}$ given by Θ plus one extra first row of length s_1. The global $\mathbb{N}$-grading of $\mathfrak{R}_0$ is given by the one-to-one map $g : \mathfrak{R} \mapsto \mathbb{N}$ defined by $g(\Theta_{[p_I];\alpha}) = \alpha + s_1$. It has the property that if $g(\Theta_\alpha) > g(\Theta_\beta)$ then $p_\alpha \leqslant p_\beta$ and $|\Theta_\alpha| > |\Theta_\beta|$.

The representation of $\mathfrak{g}_0$ in $\mathfrak{R}_q$ takes the form

$$
\rho_q = \begin{bmatrix}
(\rho_q)^{\mathbf{P}_B+\mathbf{q}}_{\mathbf{P}_B+\mathbf{q}} & (\rho_q)^{\mathbf{P}_B+\mathbf{q}}_{\mathbf{P}_{B-1}+\mathbf{q}}(e) & 0 & \cdots\cdots\cdots\cdots\cdots\cdots \\
0 & (\rho_q)^{\mathbf{P}_{B-1}+\mathbf{q}}_{\mathbf{P}_{B-1}+\mathbf{q}} & (\rho_q)^{\mathbf{P}_{B-1}+\mathbf{q}}_{\mathbf{P}_{B-2}+\mathbf{q}}(e) & 0 & \cdots\cdots \\
\vdots & 0 & (\rho_q)^{\mathbf{P}_{B-2}+\mathbf{q}}_{\mathbf{P}_{B-2}+\mathbf{q}} & (\rho_q)^{\mathbf{P}_{B-2}+\mathbf{q}}_{\mathbf{P}_{B-3}+\mathbf{q}}(e) & 0 & \cdots\cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} , \quad (2.14)
$$

where the diagonal blocks are e^a-independent representations on submodules and the off-diagonal blocks are e^a-dependent Chevalley-Eilenberg cocycles that are activated by the translations $P_a \in \mathfrak{iso}(1, D-1)$ and depend on q via phase factors. The representations of P_a within the submodules read ($\alpha \equiv k_I + s_{I+1,1}$, $k_I = 0, \ldots, s_{I,I+1} - 1$)

$$
\xi^a \left[(\rho_q)^{\mathbf{P}_I+\mathbf{q}}_{\mathbf{P}_I+\mathbf{q}}(P_a)\right]^\alpha_\beta X_q^{\mathbf{P}_I+\mathbf{q}}(\Theta^{*\beta}_{[p_I]}) = \begin{cases} \xi_{(p_I+1)} X_q^{\mathbf{P}_I+\mathbf{q}}(\Theta^{*\,\alpha+1}_{[p_I]}) & \text{if } k_I < s_{I,I+1} - 1 \\ 0 & \text{if } k_I = s_{I,I+1} - 1 \end{cases} , \quad (2.15)
$$

where $\xi_{(i)}$ denotes the operation of contracting ξ^a into the ith row of a tensor followed by Young projection onto the shape with one less cell in that row. In terms of this operation, the Chevalley-Eilenberg cocycles have representation matrices ($\alpha \equiv k_I + s_{I+1,1}$, $k_I = 0, \ldots, s_{I,I+1} - 1$)

$$
\xi^a \left[(\rho_q)^{\mathbf{P}_I+\mathbf{q}}_{\mathbf{P}_{I-1}+\mathbf{q}}(P_a|e)\right]^\alpha_\beta X_q^{\mathbf{P}_I+\mathbf{q}}(\Theta^{*\beta}_{[p_{I-1}]})
$$
$$
= \begin{cases} 0 & \text{if } k_I < s_{I,I+1} - 1 \\ (-1)^{q(h_I+1)} e_{(p_{I-1}+1)} \cdots e_{(p_I)} \xi_{(p_I+1)} X_q^{\mathbf{P}_{I-1}+\mathbf{q}}(\Theta^{*\,s_{I,1}}_{[p_{I-1}]}) & \text{if } k_I = s_{I,I+1} - 1 \end{cases} . \quad (2.16)
$$

Integrating the above representation matrix and setting the integration constant to zero, yields the operator $\sigma_q^- : \mathfrak{R}_q \to \mathfrak{R}_{q+1}$ given by

$$
\sigma_q^- = -i \int_0^1 dt \, e^a \rho_q(P_a|te) , \quad (2.17)
$$

with the following key property ($\nabla := d - \frac{i}{2}\omega^{ab}\rho(M_{ab})$ and $\nabla e^a = 0$)

$$
(\nabla + \sigma_{q+1}^-)(\nabla + \sigma_q^-) \equiv 0 \quad \Leftrightarrow \quad \nabla^2 \equiv \nabla \sigma_q^- + \sigma_{q+1}^- \nabla \equiv \sigma_{q+1}^- \sigma_q^- \equiv 0, \quad (2.18)
$$

which is equivalent to the closure of the $\mathfrak{g}_0$-transformations

$$
\delta_{\xi,\Lambda} X^\alpha = \frac{i}{2} \Lambda^{ab} \rho_q(M_{ab}) X^\alpha + i\xi^a \left[\rho_q(P_a|e)\right]^\alpha_{\alpha+1} X^{\alpha+1}, \quad \delta_{\xi,\Lambda}(e+\omega) = 0 . \quad (2.19)
$$

The Skvortsov equations are now the generalized curvature constraints

$$
R^\alpha := \left[(\nabla + \sigma_0^-)X\right]^\alpha = \nabla X^\alpha + \left[\sigma_0^-\right]^\alpha_{\alpha+1} X^{\alpha+1} \approx 0 . \quad (2.20)
$$

The first levels of Bianchi identities and gauge transformations take the form

$$Z^\alpha := [(\nabla + \sigma_1^-)R]^\alpha = \nabla R^\alpha + \left[\sigma_1^-\right]_{\alpha+1}^\alpha R^{\alpha+1} \equiv 0 \,, \tag{2.21}$$

$$\delta_\epsilon X^\alpha := [(\nabla + \sigma_{-1}^-)\epsilon]^\alpha = \nabla\epsilon^\alpha + \left[\sigma_{-1}^-\right]_{\alpha+1}^\alpha \epsilon^{\alpha+1} \,. \tag{2.22}$$

The cohomology of σ^- in the triangular module $\mathfrak{T}$ determines the on-shell content of the Skvortsov equations. In particular, the non-trivial content of $H^*(\sigma^-|\mathfrak{T}) \cap \mathfrak{R}_0$ is the (trace-constrained) Labastida gauge field

$$\varphi(\Theta) = \mathbb{P}_\Theta^{\mathfrak{sl}(D)} \, i_{\theta^{a_1}} \cdots i_{\theta^{a_H}} X^{\mathbf{P}_B}(\widetilde{\Theta}^*) \,. \tag{2.23}$$

The Labastida field equation is the non-trivial content of $H^*(\sigma^-|\mathfrak{T}) \cap \mathfrak{R}_1$. The restriction of the triangular module $\mathfrak{T}$ to its submodule $\mathfrak{T}_{\mathrm{Weyl}}$ consisting of states with $p_\alpha = 0$ yields the primary Weyl tensor $C(\overline{\Theta})$ as the non-trivial content of $H^*(\sigma^-|\mathfrak{T}_{\mathrm{Weyl}}) \cap \mathfrak{R}_0^0$.

As realized early in [34] (see [35, 36] for reviews) and also pointed out later in [21, 22], the local degrees of freedom are encoded in the Weyl zero-form module and may be put in correspondence with the massless $\mathfrak{g}_0$-irrep $\mathfrak{D}(M^2=0;\Theta)$ through harmonic expansion. Thus, for the purpose of counting the local on-shell degrees of freedom carried by $\varphi(\Theta)$ it suffices to analyze $C(\overline{\Theta})$ and it is not necessary to actually extract the precise form of the Labastida operator.

2.2 Interlude: oscillator realization of the Young tableaux

In order to study the integrability of Skvortsov's equations and more generally to describe tensor fields of arbitrary shapes, one may adopt the notion of a generalized Schur module and related hyperform complex [23, 37–40] and to give these an explicit oscillator realization [4, 5]

The general properties of the cell operators presented in section 2.2.2 suffice for handling the unfolded master-field equations in flat spacetime as well as the generic massive master-field equations in AdS_D. However, in order to examine the critically massless cases in AdS_D (namely in analyzing the projection (3.72) of the reducible Weyl zero-form) it appears that a more explicit expression for the cell operators is needed as was realized by Metsaev [4, 5]. Such an expression is rederived here and will be crucial to our analysis in section 3 — more precisely, for our derivation of (3.74).

2.2.1 Howe duality and Schur states

The decomposition of tensor products of finite-dimensional representations of the classical matrix algebras, $\mathfrak{m}$ say, using manifestly symmetric $(+)$ and anti-symmetric $(-)$ bases leads to the notion of Howe dual algebras $\widetilde{\mathfrak{m}}^\pm$ and associated generalized Schur modules $\mathcal{S}^\pm$ as described in appendix B. Using bosonic $(+)$ and fermionic $(-)$ oscillator realizations, the Lie algebra $\widetilde{\mathfrak{m}}^\pm$ arises as a subalgebra of the infinite-dimensional Lie algebra of canonical transformations of the oscillator algebra and is identified with the maximal finite-dimensional subalgebra that commutes with $\mathfrak{m}$. The corresponding $\mathcal{S}^\pm$ are by definition the subspaces

of the Fock modules $\mathcal{F}^{\pm}$ consisting of states $|\Delta\rangle^{\pm}$ that are annihilated by a Borel subalgebra of $\widetilde{\mathfrak{m}}^{\pm}$. Using $\nu_{\pm}$ oscillator flavors, say $\{\alpha_i^a, \bar{\alpha}_a^i\}_{i=1}^{\nu_{\pm}}$, leads to finite-dimensional Howe dual algebras, namely $\mathfrak{sl}(\nu_{\pm})$ for $\mathfrak{sl}(D)$ tensors, and $\mathfrak{sp}(2\nu_+)$ and $\mathfrak{so}(2\nu_-)$ for $\mathfrak{so}(D)$.

If $\mathfrak{m} = \mathfrak{sl}(D)$ then the Schur states can be chosen to obey

$$(N_j^i - \delta_j^i \widetilde{\lambda}_i^{\pm})|\Delta\rangle^{\pm} = 0, \quad 1 \leqslant i \leqslant j \leqslant \nu_{\pm}, \tag{2.24}$$

where $N_j^i \in \mathfrak{sl}(\nu_{\pm})$. If $\mathfrak{m} = \mathfrak{so}(D)$ then the Schur states also obey the tracelessness condition

$$\mathfrak{m} = \mathfrak{so}(D) : \begin{cases} T_{(11)}|\Delta\rangle^+ &= 0 \text{ in } \mathcal{F}^+ \\ T_{[12]}|\Delta\rangle^- &= 0 \text{ in } \mathcal{F}^- \end{cases} \tag{2.25}$$

where in a three-graded splitting (see (B.16)) $T_{(11)} \in [\mathfrak{sp}(2\nu_+)]^{(-1)}$ and $T_{[12]} [\mathfrak{so}(2\nu_-)]^{(-1)}$, taking the leading traces of Schur states such that (2.24) and (2.25) imply $T_{(ij)}|\Delta\rangle^+ = 0$ and $T_{[ij]}|\Delta\rangle^- = 0$. In both cases one can show that $\widetilde{w}_1^{\pm} \geqslant \cdots \geqslant \widetilde{w}_{\nu_{\pm}}^{\pm} \geqslant 0$ where $\widetilde{w}_i^{\pm} = \widetilde{\lambda}_i^{\pm} \mp \frac{D}{2}$, and that $|\Delta\rangle^{\pm}$ contains exactly one copy of the $\mathfrak{m}$-irrep with highest weight given by $\{\widetilde{w}_i^{\pm}\}_{i=1}^{\nu_{\pm}}$. Moreover, in the limit $\nu_{\pm} \to \infty$ arise the universal Howe-dual algebras $\widetilde{\mathfrak{m}}^{\pm} \cong \mathfrak{sl}(\infty)$ for $\mathfrak{sl}(D)$ tensors, and $\widetilde{\mathfrak{m}}^+ \cong \mathfrak{sp}(2\infty)$ and $\widetilde{\mathfrak{m}}^- \cong \mathfrak{so}(2\infty)$ for $\mathfrak{so}(D)$ tensors, such that

$$\nu^{\pm} \to \infty : \mathcal{S}^{\pm} \cong \mathcal{S}^{\mp}. \tag{2.26}$$

2.2.2 Cell operators: general definitions and properties

From now on we consider the general classical matrix algebras denoted here by $\mathfrak{m} = (\mathfrak{sl}(D), \mathfrak{so}(D), \mathfrak{sp}(D))$ and parameterized by $\epsilon(\mathfrak{m}) = (0, +1, -1)$, and use the notation of appendix B otherwise.

The oscillator formalism can be used to define the cell operators [4, 5, 37, 38] $\{\beta_{\pm(i),a}, \bar{\beta}^{\pm(i),a}\}_{i=1}^{\nu_{\pm}}$ as a set of operators on the oscillator module $\mathcal{M}^{\pm}$ that induces a non-trivial and regular action on the corresponding Schur modules $\mathcal{S}^{\pm}$ obeying:

(i) the amputation and generation properties

$$(N_j^i - \delta_j^i(\widetilde{\lambda}_i^{\pm} - 1))\beta_{\pm(i),a}|\Delta\rangle = 0, \quad (N_j^i - \delta_j^i(\widetilde{\lambda}_i^{\pm} + 1))\bar{\beta}^{\pm(i),a}|\Delta\rangle = 0, \quad 1 \leqslant i \leqslant j \leqslant \nu_{\pm} \tag{2.27}$$

for $|\Delta\rangle \in \mathcal{S}^{\pm}$; and

(ii) the conjugation rule

$$\bar{\beta}^{\pm(i),a} = \pi\left(\beta_{\pm(\nu_{\pm}-i+1),a}\right), \tag{2.28}$$

where $\pi := \pi_{(\nu_{\pm},\ldots,1)}$ with $(\nu_{\pm},\ldots,1)$ denoting the reverse permutation in $S_{\nu_{\pm}}$ and π_{σ} $(\sigma \in S_{\nu_{\pm}})$ being the linear automorphisms of the oscillator algebra defined for arbitrary composite operators f and g by

$$\pi_{\sigma}(fg) = \pi_{\sigma}(f)\pi_{\sigma}(g), \qquad \pi_{\sigma}\left(f(\alpha_{i,a}, \bar{\alpha}^{j,b})\right) = f(\bar{\alpha}^{\sigma(i),a}, \mp\alpha_{\sigma(j),b}). \tag{2.29}$$

The amputation property amounts to that

$$N_j^i \, \beta_{\pm(k),a} = \sum_{m<n} \gamma_{j,k,m}^{i,n} N_n^m \quad \text{if } i < j \,, \tag{2.30}$$

for some operators $\gamma_{j,k,m}^{i,n}$. Thus, the conjugation rule (2.28) is well-defined since

$$\pi_\sigma(N_j^i) = \mp N_{\sigma(i)}^{\sigma(j)} \,, \tag{2.31}$$

together with (2.30) imply that if $i < j$ then

$$\begin{aligned}
N_j^i \, \pi(\beta_{\pm(k),a})|\Delta\rangle &= \mp \pi \left(N_{\nu_\pm-i+1}^{\nu_\pm-j+1} \beta_{\pm(k),a} \right) |\Delta\rangle \\
&= \sum_{m<n} \pi \left(\gamma_{\nu_\pm-i+1,k,m}^{\nu_\pm-j+1,n} \right) N_{\nu_\pm-m+1}^{\nu_\pm-n+1} |\Delta\rangle = 0 \,.
\end{aligned} \tag{2.32}$$

The amputation and generation properties imply that

$$\beta_{\pm(i),a}|\Delta\rangle = 0 = \bar{\beta}^{\pm(i+1),a}|\Delta\rangle \qquad \text{if } \widetilde{w}_i^\pm = \widetilde{w}_{i+1}^\pm \,. \tag{2.33}$$

In the Fock space realization, where the Schur modules decompose into Young tableaux, this means that $\beta_{\pm(i),a}$ and $\bar{\beta}^{\pm(i),a}$, respectively, add and remove cells from the ith row $(+)$ or column $(-)$ of Δ in accordance with row and column order, viz.

$$\beta_{\pm(i),a}|\Delta\rangle = 0 = \bar{\beta}^{\pm(i+1),a}|\Delta\rangle \qquad \text{if } \begin{cases} w_i = w_{i+1} \quad +\,, \\ h_i = h_{i+1} \quad -\,. \end{cases} \tag{2.34}$$

[See under (B.32)–(B.34) for the definitions of w_i and h_i.] Thus, the Schur modules $\mathcal{S}_{D;\nu_\pm}^\pm \subset \mathcal{F}_{D;\nu_\pm}^\pm$ are generated by acting on $|0\rangle$ with row-ordered $(+)$ or column-ordered $(-)$ strings of $\bar{\beta}^{\pm(i),a}$ operators that in addition need to be taken to be J-traceless (see appendix B) if $\epsilon(\mathfrak{m}) = \pm 1$.

Conversely, if

$$\mathcal{O}_\xi^\pm = \xi^{a_1^1, a_2^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_r}^r} \prod_{\ell=1}^r \prod_{j=1}^{m_\ell} \beta_{\pm(i_\ell), a_j^\ell} \,, \tag{2.35}$$

where ξ is a reducible tensor of rank R and the product is left-ordered, then

$$\mathcal{O}_\xi^\pm|\Delta\rangle = |\Delta'\rangle \,, \qquad \widetilde{w}_i^{\pm\prime} = \widetilde{w}_i^\pm - \sum_{\ell=1}^r m_\ell \, \delta_{i_\ell, i} \,. \tag{2.36}$$

The amputation property implies that $\mathcal{O}_\xi^\pm$ preserves J-tracelessness in case $\epsilon(\mathfrak{m}) = \pm 1$, i.e.[2]

$$T_{ij}|\Delta\rangle = 0 \quad \Rightarrow \quad T_{ij}\beta_{\pm(k),a}|\Delta\rangle = 0 \,. \tag{2.37}$$

___

[2]Eq. (2.37) can also be checked directly using the explicit expressions (2.59) and (2.60) for $\beta_{\pm(i),a}$. The latter actually imply the stronger property $[T_{11}, \beta_{\pm(i),a}] = 0$ in the case $\epsilon = +1$.

One can decompose the tensor ξ into irreducible representations

$$\xi^{a_1^1, a_2^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_r}^r} = \sum_{\widetilde{\Delta}} \sum_{\tau_{\widetilde{\Delta}}} (\mathbb{P}_{\tau_{\widetilde{\Delta}}} \xi)^{a_1^1, a_2^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_r}^r} , \quad (2.38)$$

where we sum over the different inequivalent Young tableaux $\tau_{\widetilde{\Delta}}$ with rank-R shape $\widetilde{\Delta}$. Thus

$$|\Delta'\rangle = \sum_{\Delta_\xi \in \Delta/\Delta'} \mathcal{O}^{\pm}_{\Delta_\xi} |\Delta\rangle , \quad (2.39)$$

where $\mathcal{O}^{\pm}_{\Delta_\xi}$ gathers together the contribution to $\mathcal{O}^{\pm}_\xi$ from all inequivalent tableaux τ_{Δ_ξ} corresponding to the diagram Δ_ξ, and Δ/Δ' is the set of Young diagrams Δ_ξ of rank R such that the outer product $\Delta_\xi \otimes \Delta'$ contains Δ with multiplicity $\text{mult}(\Delta|(\Delta_\xi \otimes \Delta')) \geqslant 1$. More precisely, $\mathcal{O}^{\pm}_{\Delta_\xi} |\Delta\rangle$ is the sum

$$\mathcal{O}^{\pm}_{\Delta_\xi} |\Delta\rangle = \sum_{\tau_{\Delta_\xi}} (\mathbb{P}_{\tau_{\Delta_\xi}} \xi)^{a_1^1, a_2^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_r}^r} \prod_{\ell=1}^{r} \prod_{j=1}^{m_\ell} \beta_{\pm(i_\ell), a_j^\ell} |\Delta\rangle \quad (2.40)$$

so that one has

$$|\Delta'\rangle = \left(\sum_{\Delta_\xi \in \Delta/\Delta'} \sum_{\tau_{\Delta_\xi}} (\mathbb{P}_{\tau_{\Delta_\xi}} \xi)^{a_1^1, a_2^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_r}^r} \right) \left[\prod_{\ell=1}^{r} \prod_{j=1}^{m_\ell} \beta_{\pm(i_\ell), a_j^\ell} |\Delta\rangle \right] . \quad (2.41)$$

Depending on the symmetries of $|\Delta\rangle$ not all the *diagrams* Δ_ξ need to contribute to the above expression. However, if one diagram Δ_ξ contributes nontrivially, then all the tableaux τ_{Δ_ξ} will contribute if ξ has no definite symmetry property. If ξ already possesses some symmetry properties in some of its indices, then several tableaux with the same shape will give the same contributions (up to an overall coefficient).

A special case, which ensures the integrability of the various master-field equations, is when $R = m + n$ with $\beta^a_{\pm(i)}$ appearing twice, so that the sequence of cell operators is as follows

$$\beta^{b_1}_{\pm(i-n+1)} \beta^{b_2}_{\pm(i-n+2)} \cdots \beta^{b_{n-1}}_{\pm(i-1)} \beta^{b_n}_{\pm(i)} \beta^{a_1}_{\pm(i)} \beta^{a_2}_{\pm(i+1)} \cdots \beta^{a_{m-1}}_{\pm(i+m-2)} \beta^{a_m}_{\pm(i+m-1)} ,$$

and when Δ contains a block of height $h = n$ between the $(i-n+1)$th and the ith rows on top of a block of height $h' = m-1$ between the $(i+1)$th and the $(i+m-1)$th rows. Then

$$\{\Delta_\xi\} = \{[m+n-p, p]\}_{p=1}^{\min(m,n)} , \quad (2.42)$$

where we note that $\text{mult}(\Delta|(\Delta' \otimes [m+n-1, 1])) = 1$ (higher multiplicities arise for $p \geqslant 2$). For $m = n = 1$, the above reduces to

$$[\beta^{\pm(i),a}, \beta^{\pm(i),b}] = 0 , \qquad [\bar{\beta}^{\pm(i),a}, \bar{\beta}^{\pm(i),b}] = 0 , \quad (2.43)$$

and for $m = 1$ and $h = n > 1$, $|\Delta\rangle$ containing a block of height h between the $(i-h+1)$th and ith rows, then

$$\prod_{\ell=1}^{h} \beta^{a_\ell}_{(i-h+l)} |\Delta\rangle = \beta^{[a_1}_{(i-h+1)} \cdots \beta^{a_h]}_{(i)} |\Delta\rangle \quad (2.44)$$

and only the two-column diagram $\Delta_\xi = [h, 1]$ will contribute to $|\Delta'\rangle$.

We note that (2.27) together with with the Casimir formula (B.25) yield

$$[C_2[\mathfrak{m}], \beta_{\pm(k),a}] = (-D + \epsilon \mp (2N_k^k + 2 - 2k))\beta_{\pm(k),a}, \qquad (2.45)$$

$$[C_2[\mathfrak{m}], \bar\beta^{\pm(k),a}] = (D - \epsilon \pm (2N_k^k - 2k))\bar\beta^{\pm(k),a}. \qquad (2.46)$$

In the case of $\epsilon = \pm 1$, these commutators imply the following anti-commutators:

$$\{M_{ac}, \beta_{\pm(k)}^c\} = i[C_2[\mathfrak{m}], \beta_{\pm(k),a}] = i(-D + \epsilon \mp (2N_k^k + 2 - 2k))\beta_{\pm(k),a}, \quad (2.47)$$

$$\{M_{ac}, \bar\beta^{\pm(k),c}\} = i[C_2[\mathfrak{m}], \bar\beta_a^{\pm(k)}] = i(D - \epsilon \pm (2N_k^k - 2k))\bar\beta_a^{\pm(k)}. \quad (2.48)$$

They also imply that

$$\bar\beta^{\pm(i),a}\beta_{\pm(j),a} = \beta_{\pm(j),a}\bar\beta^{\pm(i),a} = 0 \quad \text{if } i \neq j. \qquad (2.49)$$

The solution space to (2.27) and (2.28) is invariant under rescalings of the form

$$\beta_{\pm(i),a} \rightarrow \beta_{\pm(i),a}\, f_{\pm(i)} = t_1(i)f_{\pm(i)}\, \beta_{\pm(i),a}, \qquad (2.50)$$

where $f_{\pm(i)} = f_{\pm(i)}(N_1^1, \ldots, N_{\nu_\pm}^{\nu_\pm})$ are functions that are regular and non-vanishing on $\mathcal{S}^\pm$, and we use the notation

$$t_x(i)f := f(\ldots, N_i^i + x, \ldots) \quad \text{for } f = f(N_1^1, \ldots, N_{\nu_\pm}^{\nu_\pm}). \qquad (2.51)$$

This ambiguity can be removed partially by considering normalized cell operators $(\gamma_{\pm(i),a}, \bar\gamma^{\pm(i),a})$ obeying $\sum_{i=1}^{\nu_\pm}[\gamma_{\pm(i),a}, \bar\gamma^{\pm(i),a}] = M_b^a$ which fixes the scale factors up to constant rescalings at least for $\nu = 2$. However, at the level of the free master-field equations, the normalization is immaterial since the rescalings (2.50) amount to non-singular redefinitions of auxiliary fields.

2.2.3 Cell operators: explicit oscillator realization

The explicit form of the cell operators can be found by an iterative procedure based on the assumption that $\beta_{\pm(i),a}$ only depends on $\alpha_{j,a}$ and N_k^j with $j \geqslant k \geqslant i$. Then $N_k^j\beta_{\pm(i),a}|\Delta\rangle = 0$ for $j < k < i$ and it remains to solve $N_k^j\beta_{\pm(i),a}|\Delta\rangle = 0$ for $i \leqslant j < k \leqslant \nu_\pm$. From $N_k^j\beta_{\pm(i+1),a}|\Delta\rangle = 0$ for $i + 1 \leqslant j \leqslant \nu_\pm$ it follows that $N_k^j\check\beta_{\pm(i),a}|\Delta\rangle = 0$ for $i \leqslant j \leqslant \nu_\pm - 1$ where $\check\beta_{\pm(i),a} = \beta_{\pm(i),a}|_{(\alpha_{i',a}, \bar\alpha^{i',a}) \rightarrow (\alpha_{i'-1,a}, \bar\alpha^{i'-1,a})}$. Thus, by the assumption,

$$\beta_{\pm(i),a} = \check\beta_{\pm(i),a}g_{(i)} + \alpha_{\nu_\pm,a}N_i^{\nu_\pm}f_{(i,\nu_\pm)}$$
$$+ \sum_{p=1}^{\nu_\pm - i - 1}\sum_{i < j_1 \cdots < j_p < \nu_\pm} \alpha_{\nu_\pm,a}N_{j_p}^{\nu_\pm}\cdots N_i^{j_1}f_{(i,j_1,\ldots,j_p,\nu_\pm)} + \cdots,$$

where $g_{(i)}$, $f_{(i,\nu_\pm)}$ and $f_{(i,j_1,\ldots,j_p,\nu_\pm)}$ are functions of $(N_i^i, \ldots, N_{\nu_\pm}^{\nu_\pm})$ to be determined from

$$N_{j+1}^j\beta_{\pm(i),a} = 0 \qquad \text{for } j = i, \ldots, \nu_\pm - 1, \qquad (2.52)$$

$$-\ 10\ -$$

and the initial condition

$$\beta_{\pm(\nu_\pm),a} = \alpha_{\nu_\pm,a} \,. \tag{2.53}$$

One solution, which is actually regular on $\mathcal{M}^\pm$, is

$$g_{(i)} \simeq P(i,\nu_\pm)\,, \quad f_{(i,\nu_\pm)} \;\simeq\; \frac{\prod_{j=i+1}^{\nu_\pm} P(i,j)}{P(i,\nu_\pm)}\,,$$

$$f_{(i,j_1,\ldots,j_p,\nu_\pm)} \;\simeq\; \frac{\prod_{j=i+1}^{\nu_\pm} P(i,j)}{\prod_{q=1}^{p} P(i,j_q)}\,, \tag{2.54}$$

that is

$$\beta_{\pm(i),a} \simeq \left[\alpha_{i,a} + \sum_{i<j_1<\cdots<j_p\leqslant\nu_\pm} \alpha_{j_p,a} N_{j_{p-1}}^{j_p} \cdots N_i^{j_1} \right] \frac{\prod_{j=i+1}^{\nu_\pm} P(i,j)}{\prod_{q=1}^{p} P(i,j_q)}$$

$$= \left[\alpha_{i,a} + \sum_{i<j_1<\cdots<j_p\leqslant\nu_\pm} N_i^{j_1} \frac{1}{P(i,j_1)} \cdots N_{j_{p-1}}^{j_p} \frac{1}{P(i,j_p)} \alpha_{j_p,a} \right] \prod_{j=i+1}^{\nu_\pm} (P(i,j)+1) \,, \tag{2.55}$$

where $\simeq$ refers to the ambiguity residing in rescalings of the form (2.50), and

$$P(i,j) = N(i,j) + j - i - 1\,, \qquad N(i,j) \;=\; N_i^i - N_j^j \,. \tag{2.56}$$

Correspondingly,

$$\bar\beta^{\pm(i),a} \simeq \pi(\beta_{\pm(\nu_\pm-i+1),a}) \tag{2.57}$$

$$= \left[\bar\alpha^{i,a} + \sum_{1\leqslant j_p<\cdots<j_1<i} (-1)^p N_{j_1}^i \frac{1}{P(j_1,i)} \cdots N_{j_p}^{j_{p-1}} \frac{1}{P(j_p,i)} \bar\alpha^{j_p,a} \right] \prod_{j=1}^{i-1} (P(j,i)+1) \tag{2.58}$$

The overall factors $\prod_{j=i+1}^{\nu_\pm}(P(i,j)+1)$ and $\prod_{j=1}^{i-1}(P(j,i)+1)$ as well as the inverses of $P(i,j)$ and $P(j,i)$ are regular and non-vanishing in $\mathcal{S}^\pm$. Thus, as long as regularity in $\mathcal{M}$ is not of any concern, one may rescale the cell operators, and work with

$$\beta_{\pm(i),a} = \alpha_{i,a} + \sum_{i<j_1\cdots<j_p\leqslant\nu_\pm} N_i^{j_1} \frac{1}{P(i,j_1)} \cdots N_{j_{p-1}}^{j_p} \frac{1}{P(i,j_p)} \alpha_{j_p,a} \,, \tag{2.59}$$

$$\bar\beta^{\pm(i),a} = \bar\alpha^{i,a} + \sum_{1\leqslant j_p<\cdots<j_1<i} (-1)^p N_{j_1}^i \frac{1}{P(j_1,i)} \cdots N_{j_p}^{j_{p-1}} \frac{1}{P(j_p,i)} \bar\alpha^{j_p,a} \,. \tag{2.60}$$

2.2.4 Equivalent bosonic and fermionic universal Schur modules

By definition, the Fock spaces $\mathcal{F}_{D;\nu_\pm}^\pm$ and the corresponding generalized Schur modules $\mathcal{S}_{D;\nu_\pm}^\pm \subset \mathcal{F}_{D;\nu_\pm}^\pm$ consist of states $|\Psi\rangle = \Psi(\bar\alpha^{i,a})|0\rangle$ and $|\Delta\rangle = \Delta(\bar\alpha^{i,a})|0\rangle$, respectively, generated by $\Psi(\bar\alpha^{i,a})$ and $\Delta(\bar\alpha^{i,a})$ that are arbitrary polynomials. Acting on $|\Delta\rangle$ with the cell operators $\beta_{\pm(i),a}|\Delta\rangle$ and $\bar\beta^{\pm(i),a}|\Delta\rangle$, given in (2.59) and (2.60), yields states $\beta_{\pm(i),a}|\Delta\rangle$ and $\bar\beta^{\pm(i),a}|\Delta\rangle$ that remain arbitrary polynomials (finite sums) for arbitrary $\nu_\pm$. Thus, the cell operators have a well-defined action in $\mathcal{S}_{D;\nu_\pm}^\pm$ in the limit $\nu_\pm \to \infty$. From the

– 11 –

expressions (B.38), (B.39) and (B.58) for the multiplicities, it follows that the bosonic and fermionic oscillator realizations are on equal footing in the sense that

$$\mathcal{S}^+_{D;\nu_+} \simeq \mathcal{S}^-_{D;\infty} \quad \text{for } \nu_+ \geqslant D \,, \tag{2.61}$$

and, taking into account the fact that $h_{w_i} \geqslant i$ and $h_{w_{i+1}} \leqslant i - 1$, one finds

$$\beta_{(i),a}|\Delta\rangle \simeq \sum_{j=1}^{\infty} \beta_{[j],a}\delta_{h_j,i}|\Delta\rangle \,, \qquad \bar{\beta}^{(i),a}|\Delta\rangle \simeq \sum_{j=1}^{\infty} \bar{\beta}^{[j+1],a}\delta_{h_j,i-1}|\Delta\rangle \,, \tag{2.62}$$

where we use the notation

$$\beta_{(i),a} := \beta_{+(i),a} \,, \qquad \beta_{[i],a} := \beta_{-(i),a} \,, \tag{2.63}$$

idem $\bar{\beta}$.

For example, for $D = 1$ the ground states are $|\Delta\rangle = |(n)\rangle = |[\ \underbrace{1,\ldots,1}_{n \text{ columns}}\ ]\rangle$, and

$$\beta_{[i]}|(n)\rangle = \delta_{in}|(n-1)\rangle \,, \qquad \bar{\beta}^{[i]}|(n)\rangle = \delta_{i,n+1}|(n+1)\rangle \,, \tag{2.64}$$

and the above map takes the form

$$\beta := \beta_{(1)} = \sum_{i=1}^{\infty} \beta_{[i]} = \alpha\frac{1}{\sqrt{\bar{\alpha}\alpha}} \,, \qquad \bar{\beta} := \bar{\beta}^{(1)} = \sum_{i=1}^{\infty} \bar{\beta}^{[i]} = \frac{1}{\sqrt{\bar{\alpha}\alpha}}\bar{\alpha} \,, \tag{2.65}$$

where $\alpha := \alpha_1$ and $\bar{\alpha} := \bar{\alpha}^1$ obey $[\alpha, \bar{\alpha}] = 1$ and we note that $\{\beta, \bar{\beta}\} = 1$.

Roughly speaking, the correspondence between the bosonic and fermionic oscillators is the result of "gauging" on the one hand $\widetilde{\mathfrak{m}}^- = \mathfrak{gl}(\infty)$ in $\mathcal{F}^-_{D;\infty}$, and on the other hand $\widetilde{\mathfrak{m}}^+ = \mathfrak{gl}(\nu_+)$ in $\mathcal{F}^+_{D;\nu_+}$ for $\nu_+ \geqslant D$. Thus, in the limit $\nu_+ \to \infty$,

$$\mathcal{S}^+_{D;\infty} \cong \mathcal{S}^-_{D;\infty} \,, \tag{2.66}$$

where both sides are $\mathfrak{gl}(\infty)$-gauged oscillator spaces.

2.3 Master-field reformulation of Skvortsov's equations

The master field

$$\mathbf{X} := \sum_{p=0}^{\infty} \mathbf{X^p} \in \mathfrak{R} = \bigoplus_{p\geqslant 0}\mathfrak{R^p} \,, \quad \mathfrak{R^p} := \Omega^{\mathbf{p}}(U) \otimes \mathcal{S} \,, \tag{2.67}$$

where $\mathcal{S}$ is the Schur module described in the previous section. The Skvortsov equations amount to subjecting $\mathbf{X}$ to: i) curvature constraints; and ii) mass-shell and irreducibility conditions. The curvatures and irreducibility conditions can be examined at the level of the $\mathfrak{sl}(D)$ Schur module, while the mass-shell condition breaks $\mathfrak{sl}(D)$ down to $\mathfrak{so}(1, D-1)$.

2.3.1 Bosonic oscillators (symmetric basis)

Curvature constraints. The generalized curvature constraints can be written using symmetric conventions as

$$\mathbf{R} := \left(\nabla + \sigma_0^-\right)\mathbf{X} \approx 0, \qquad \sigma_0^- := \sum_{p \geqslant p'}(\sigma_0)_{\mathbf{p'}}^{\mathbf{p+1}}, \tag{2.68}$$

$$(\sigma_0)_{\mathbf{p'}}^{\mathbf{p+1}} := -ie_{(p'+1)}\cdots e_{(p+1)}\mathbb{P}(p+1, p'+1), \tag{2.69}$$

where $\nabla := d - \frac{i}{2}\omega^{ab}M_{ab}$, $e_{(i)} := e^a\beta_{(i),a}$ and $\mathbb{P}(p+1, p'+1) : \mathfrak{R} \to \mathfrak{R}^{\mathbf{p}}$ is a projector defined by

$$\mathbb{P}(p+1, p'+1)\mathbf{X} := \begin{cases} \delta\left\{N(p'+1, p'+2), N(p'+2, p'+3), \ldots, N(p, p+1)\right\}\mathbf{X}^{\mathbf{p'}} & \text{for } p > p' \\ \mathbf{X}^{\mathbf{p}} & \text{for } p = p' \end{cases}, \tag{2.70}$$

where $\delta\{\lambda_1, \ldots, \lambda_k\} := \delta_{\lambda_1,0}\cdots\delta_{\lambda_k,0}$ for $\lambda_i \in \mathbb{Z}$, $i = 1, \ldots, k$. The corresponding triangular module has the generalized curvatures ($q \in \mathbb{Z}$)

$$\mathbf{Z}_{q+1} := (\nabla + \sigma_q^-)\mathbf{Z}_q, \tag{2.71}$$

$$\sigma_q^- = (-1)^{q(1+\sigma_\circ^-)}\sigma_0^- = \sum_{p \geqslant p'}(-1)^{q(p-p')}(\sigma_0)_{\mathbf{p'}}^{\mathbf{p+1}}. \tag{2.72}$$

The Cartan integrability amounts to the identity

$$0 \equiv -\mathbf{Z}_2 := -\left[(-1)^{1+\sigma_\circ^-}\sigma_0^-\right]\sigma_0^-\mathbf{X} \tag{2.73}$$

$$= \sum_{p \geqslant p' \geqslant r'}(-1)^{p-p'}e_{(p'+1)}\cdots e_{(p+1)}\mathbb{P}(p+1, p'+1)e_{(r'+1)}\cdots e_{(p'+1)}\mathbb{P}(p'+1, r'+1)\mathbf{X} \tag{2.74}$$

$$= e_{(1)}e_{(1)}\mathbf{X}^{\mathbf{0}} + \left(e_{(2)}e_{(2)}\mathbf{X}^{\mathbf{1}} + e_{(2)}e_{(1)}e_{(2)}\mathbb{P}(2,1)\mathbf{X}^{\mathbf{0}} - e_{(1)}e_{(2)}\mathbb{P}(2,1)e_{(1)}\mathbf{X}^{\mathbf{0}}\right)$$

$$+ e_{(3)}e_{(3)}\mathbf{X}^{\mathbf{2}} + e_{(3)}e_{(2)}e_{(3)}\mathbb{P}(3,2)\mathbf{X}^{\mathbf{1}} + e_{(3)}e_{(1)}e_{(2)}e_{(3)}\mathbb{P}(3,1)\mathbf{X}^{\mathbf{0}} - e_{(2)}e_{(3)}\mathbb{P}(3,2)e_{(2)}\mathbf{X}^{\mathbf{1}}$$

$$- e_{(2)}e_{(3)}\mathbb{P}(3,2)e_{(1)}e_{(2)}\mathbb{P}(2,1)\mathbf{X}^{\mathbf{0}} + e_{(1)}e_{(2)}e_{(3)}\mathbb{P}(3,1)e_{(1)}\mathbf{X}^{\mathbf{0}} + \cdots \tag{2.75}$$

where the first term is the Bianchi identity for the 0-form constraint, the second group of terms is the Bianchi identity for the 1-form constraint and last two lines is the Bianchi identity for the 2-form constraint. The terms of the form $e_{(p+1)}e_{(p+1)}\mathbf{X}^{\mathbf{p}}$ vanish by virtue of $[\beta_{(p+1),a}, \beta_{(p+1),b}] = 0$ where the commutator is induced by the anti-commutativity of e^a. The terms cubic in e^a vanish because of (2.42). For example, $\beta_{(p+2),a}\beta_{(p+1),b}\beta_{(p+2),c}\mathbb{P}(p+2, p+1)\mathbf{X}^{\mathbf{p}}$ contains a hooked Young tableau in the indices (a, b, c) on which the totally anti-symmetric projection enforced by $e^a e^b e^c$ vanishes. Similarly, $e_{(p+1)}e_{(p+2)}\mathbb{P}(p+2, p+1)e_{(p+1)}\mathbf{X}^{\mathbf{p}}$ projects on types having $w_{p+1} = w_{p+2} + 1$ (using the notation in (B.32) and (B.33)) but then, by (2.42), this results in hooked shape that gives zero. The first term in the last line, which is quartic in e^a, can be non-zero only if the types in $\mathbf{X}^{\mathbf{0}}$ have the symmetry property $w_1 = w_2 = w_3 + 1$, because of the presence of the two projectors $\mathbb{P}(2,1)$ and $\mathbb{P}(3,2)$. Likewise, the projectors in the last term enforces $w_1 = w_2 + 1 = w_3 + 1$. However, by (2.42) again, the resulting hooked shapes are incompatible with the total antisymmetry enforced by the four vielbeins.

In the general case, $\mathbb{P}(p+1, p'+1)$ and $\mathbb{P}(p'+1, r'+1)$ force the flat indices of the cell operators to be projected on different two-column Young tableaux associated with the shapes given in (2.42), where $m = p - p' + 1$ and $n = p' - r' + 1$, with maximal height $m + n - 1 = p - r' + 1$. However, there are $m + n = p - r' + 2$ vielbeins whose flat indices are to be contracted with the ones of the cell operators, which yields zero.

Mass-shell and irreducibility conditions. The mass-shell and irreducibility conditions are not unique at the free-field level. Two natural models are:

(1) the minimal trace-constrained Skvortsov system defined by

$$N_j^i \mathbf{X^P} = T_{11} \mathbf{X^P} = 0 \qquad \text{for } i < j \text{ and } \forall p \, ; \tag{2.76}$$

and

(2) the non-minimal trace-unconstrained system, viz.

$$N_i^j \mathbf{X^P} = 0 \qquad \text{for } i < j \text{ and } \forall p \, , \tag{2.77}$$

$$T_{11} \mathbf{X^0} = 0 \, . \tag{2.78}$$

Both systems carry the same physical degrees of freedom, namely one massless particle for each $\overline{\Theta}$ in $(\mathrm{Ker}N(1,2)) \cap \mathcal{S}_D$. The minimal system suffices for constructing first-order Skvortsov-Vasiliev-Weyl-type actions. The non-minimal system contains additional Stückelberg potentials that could turn out to be useful in constructing first-order actions that are equivalent to the unconstrained metric-like formulation of mixed-symmetry fields [26].

At the non-linear level, the spectrum is to be determined by some nonabelian extension of $\mathfrak{iso}(1, D-1)$. Non-linearities are also sensitive to whether the constraints are imposed strongly, as above, or weakly by means of multiplication by a projector, or more generally, by means of a suitable BRST operator.

2.3.2 Fermionic oscillators (anti-symmetric basis)

The equivalence between the bosonic and fermionic oscillator realizations of the universal Schur module discussed in section 2.2.4 can be used to cast the manifestly symmetric master-field formulation into a manifestly anti-symmetric ditto obtained by substituting

$$\beta_{a,(i)} \to \sum_{j=1}^{\infty} \beta_{a,[j]} \, \delta\left\{ N_j^j + \frac{D}{2} - i \right\} , \tag{2.79}$$

where $\delta\{\lambda\} = \delta_{\lambda,0}$ for $\lambda \in \mathbb{Z}$ and the eigenvalues of N_j^j are given by $n_j - \frac{D}{2}$ where n_j is the height of the jth column. The σ^--operator now takes the form

$$\sigma_0^- = -i \sum_{p \geqslant p'} \sum_{j_{p'}, \ldots, j_p = 1}^{\infty} \delta_{n_{j_{p'}}, p'+1} \cdots \delta_{n_{j_p}, p+1} e_{[j_{p'}]} \cdots e_{[j_p]} \mathbb{P}(p+1, p'+1) , \tag{2.80}$$

with $\mathbb{P}(p+1, p'+1)$ defined by (2.70). This expression can be rearranged into the manifestly anti-symmetric form

$$\sigma_0^- \mathbf{X} = -i \sum_{p \geqslant p'} \sum_{i=1}^{\infty} \left(e_{[i]}\right)^{p-p'+1} \delta\left\{ N_i^i + \frac{D-2}{2} - p \right\} \mathbf{X}^{\mathbf{p}'} . \qquad (2.81)$$

3 Tensor fields in AdS_D

This section contains the derivation of the unfolded equations of motion for arbitrary tensor gauge fields in AdS_D by radial reduction of Skvortsov's equations in $\mathbb{R}^{2,D-1}$. We use the master-field formulation given in section 2.3 and the foliation lemmas of section I.3.7, and follow the step-by-step procedure outlined in section I.4.5 whereby one

1) decomposes the variables and generalized curvatures into components parallel and transverse to the radial vector field;

2) constrains the radial derivatives in terms of a massive parameter f (cf. item (i) of section I.3.7);

3) shows that a generic value for $C_2[\mathfrak{g}_\lambda]$ corresponds to two "dual" values $f^{\pm}$ of f obeying $f^+ \geqslant f^-$ and $f^+ + f^- = D-1$, and that in our parametrization turn out to be $f^+ = e_0$, the lowest energy of the physical lowest-weight space, and $f^- = \widetilde{e}_0$, the lowest energy of its shadow;

4) examines the critical limit where $f = f_I^+$ approaches Metsaev's massless values e_0^I, for which we claim (and prove in a subset of all cases) that f_I^- (given by (3.74)) is consistent with a *projection* of the radially reduced Weyl zero-form onto its massless sector (cf. item (ii) of section I.3.7), whose complement thus constitutes an ideal;

5) shows that the potential module, as defined in section I.4.4.4, is trivial except in the unitary massless case $I = 1$ where it consists of the ASV potential;

6) shows the smoothness of the flat limit of the projected massless system, and how the BMV conjecture is realized in an enlarged setting with extra topological fields arising in the flat limit. The latter represent the unfolded "frozen" Stückelberg fields of the Ith block whose Weyl zero form is set to zero in the aforementioned projection of the zero-form.

3.1 Transverse and parallel components in $\mathbb{R}^{2,D-1}$

Skvortsov's equations in a flat $(D+1)$-dimensional spacetime $\widehat{\mathcal{M}}_{D+1}$ with signature $(2, D-1)$ read

$$\widehat{\mathbf{T}} := \left(\widehat{\nabla} + \widehat{\sigma}_0^-\right)\widehat{\mathbf{W}} \approx 0 \,, \quad \widehat{\sigma}_0^- := -i \sum_{p \geqslant p'} \widehat{E}_{(p'+1)} \cdots \widehat{E}_{(p+1)} \widehat{\mathbb{P}}(p+1, p'+1) \,, \quad (3.1)$$

with $\widehat{\nabla} := d - \frac{i}{2}\widehat{\Omega}^{AB}\widehat{M}_{AB}$, $\widehat{E}_{(i)} := \widehat{E}^A \widehat{\beta}_{A,(i)}$ and $\widehat{\mathbf{W}} \in \widehat{\mathfrak{R}} = \bigoplus_{p \geqslant 0} \Omega^{\mathbf{p}}\left(\widehat{U}\right) \otimes \widehat{\mathcal{S}}_{D+1}$, where $\widehat{U}$ is a region of $\widehat{\mathcal{M}}_{D+1}$ that admits a foliation with AdS_D leaves and $\widehat{\mathcal{S}}_{D+1}$ is the

generalized $\widehat{\mathfrak{m}} \cong \mathfrak{so}(2, D-1)$ Schur module consisting of all possible tensorial $\widehat{\mathfrak{m}}$-types $\widehat{\Theta}_\alpha$, each occurring with multiplicity one. In the module $\widehat{\mathcal{S}}_{D+1}$, the following relations hold true:

$$\widehat{\beta}^A_{(1)}\widehat{\beta}_{A,(1)} = 0 \,, \qquad \widehat{\xi}^B\left\{\widehat{M}_B{}^A, \widehat{\beta}_{A,(1)}\right\} = -i\,(2\widehat{N}^1_1 + D)\widehat{\xi}_{(1)} \tag{3.2}$$

where eq. (2.47) is used for the second equality.

If $\xi = \xi^M \partial_M$ denotes the radial vector field in $\widehat{U}$ obeying $\xi^2 = -1$, where $\widehat{\xi}^A := \widehat{E}^A_M \xi^M$ and $N := dL$ denotes the corresponding normal one-form, then $\widehat{E}^A = \widehat{e}^A + N\widehat{\xi}^A$ and $\widehat{\Omega}^{AB} = \widehat{\omega}^{AB} + N\widehat{\Lambda}^{AB}$ where $i_\xi \widehat{e}^A = 0 = i_\xi \widehat{\omega}^{AB}$. The local $\widehat{\mathfrak{m}}$-symmetry can be used to set $d\widehat{\xi}^A = 0$ and $\widehat{\Lambda}^{AB} = 0$, that are preserved under residual local $\mathfrak{m}$-transformations on the AdS_D leaves. As described in appendix C, the transverse components $\widehat{e}^A$ and $\omega^{AB} := \widehat{\omega}^{AB} + \lambda(\widehat{e}^A\widehat{\xi}^B - \widehat{\xi}^A\widehat{e}^B)$ then obey $\widehat{\xi}_A\widehat{e}^A = 0$ and $\widehat{\xi}_A\omega^{AB} = 0$. Thus, if $i_L : AdS_D(L) \to \widehat{U}$ denotes the embedding of the AdS_D leaf of radius $L = 1/\lambda$ into $\widehat{U}$, then the vielbein and $\mathfrak{so}(1, D-1)$-valued connection on $AdS_D(L)$ are given by $e^a := i_L^* \mathbb{P}^a_A\widehat{e}^A$ and $\omega^{ab} := i_L^* \mathbb{P}^a_A \, \mathbb{P}^b_B \omega^{AB}$, where $\mathbb{P}^a_A\widehat{\xi}^A \equiv 0$. As a result, the canonical $AdS_D(L)$ connection $i_L^*\widehat{\omega}^{AB} =: \Omega^{AB} = (\omega^{ab}, \lambda e^a)$ obeys $d\Omega^{AB} + \Omega^{AC}\Omega_C{}^B = 0$, that is, $\nabla e^a = 0$ and $d\omega^{ab} + \omega^{ac}\omega_c{}^b + \lambda^2 e^a e^b = 0$. The radial reduction can also be analyzed directly on $\widehat{U}$, where one has

$$\widehat{\nabla} = d - \frac{i}{2}\widehat{\omega}^{AB}\widehat{M}_{AB} \,, \quad \widehat{E}_{(i)} = \widehat{e}_{(i)} + N\widehat{\xi}_{(i)} \,, \quad \widehat{\nabla}\widehat{E}_{(i)} = 0 \,, \qquad \widehat{\nabla}^2 = 0 \,, \tag{3.3}$$

$$\widehat{\nabla}\widehat{e}_{(i)} = \lambda N\widehat{e}_{(i)} \,, \qquad\qquad \widehat{\nabla}\widehat{\xi}_{(i)} = \lambda\widehat{e}_{(i)} \,, \qquad\qquad \widehat{\nabla}\lambda = -\lambda^2 N \,. \tag{3.4}$$

The foliation also induces a splitting of $\widehat{\mathbf{W}}$ into transverse and parallel components, say

$$\widehat{\mathbf{W}}^{\mathbf{p}} := \widehat{\mathbf{X}}^{\mathbf{p}} + N\,\widehat{\mathbf{Y}}^{\mathbf{p-1}} \in \mathfrak{R}_\perp \oplus \mathfrak{R}_\| \,, \tag{3.5}$$

$$i_\xi \widehat{\mathbf{X}}^{\mathbf{p}} := 0 \,, \qquad\qquad\qquad i_\xi \widehat{\mathbf{Y}}^{\mathbf{p-1}} := 0 \,, \tag{3.6}$$

and a corresponding decomposition $\widehat{\mathbf{T}} = \widehat{\mathbf{R}} + N\,\widehat{\mathbf{S}}$ where $i_\xi\widehat{\mathbf{R}} := 0$ and $i_\xi\widehat{\mathbf{S}} := 0$.
It follows that

$$\widehat{\mathbf{R}}^{\mathbf{p+1}} = \left(\widehat{\nabla} - N\mathcal{L}_\xi - i\,\widehat{e}_{(p+1)}\right)\widehat{\mathbf{X}}^{\mathbf{p}} + \sum_{p \geqslant p'+1}(\widehat{\sigma}_0^-)^{\mathbf{p+1}}_{\mathbf{p'}}\widehat{\mathbf{X}}^{\mathbf{p'}} \,, \tag{3.7}$$

$$\widehat{\mathbf{S}}^{\mathbf{p}} = \left(\widehat{\nabla} - N\mathcal{L}_\xi - i\,\widehat{e}_{(p+1)}\right)\widehat{\mathbf{Y}}^{\mathbf{p-1}} + \widehat{\mathbf{Z}}^{\mathbf{p}} + \sum_{p \geqslant p'+1}(-1)^{p-p'}(\widehat{\sigma}_0^-)^{\mathbf{p+1}}_{\mathbf{p'}}\widehat{\mathbf{Y}}^{\mathbf{p'-1}} \,, \tag{3.8}$$

where $(\widehat{\sigma}_0^-)^{\mathbf{p+1}}_{\mathbf{p'}} := -i\,\widehat{e}_{(p'+1)}\cdots\widehat{e}_{(p+1)}\,\widehat{\mathbb{P}}(p+1, p'+1)$ and $(p \geqslant 1)$

$$\widehat{\mathbf{Z}}^{\mathbf{p}} := (-\mathcal{L}_\xi + i\widehat{\xi}_{(p+1)})\widehat{\mathbf{X}}^{\mathbf{p}}$$
$$+ i\sum_{p \geqslant p'+1}(p - p' + 1)\widehat{\xi}_{(p'+1)}\widehat{e}_{(p'+2)}\cdots\widehat{e}_{(p+1)}\widehat{\mathbb{P}}(p+1, p'+1)\widehat{\mathbf{X}}^{\mathbf{p'}} \,. \tag{3.9}$$

3.2 Radial reduction

3.2.1 Radial Lie derivatives and unfolded mass terms

Upon constraining the radial derivatives to be scaling dimensions, i.e.

$$(\mathcal{L}_\xi + \lambda\Delta_{[p]})\widehat{\mathbf{X}}^{\mathbf{p}} \approx 0 \,, \qquad (\mathcal{L}_\xi + \lambda\Upsilon_{[p]})\widehat{\mathbf{Y}}^{\mathbf{p-1}} \approx 0 \,, \tag{3.10}$$

where $\Delta_{[p]} = \Delta_{[p]}(\{\widehat{N}_i^i\}_{i=1}^{\nu})$ idem $\Upsilon_{[p]}$, the reduced curvatures $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{S}}$ form a closed subsystem with variables $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$. Its Cartan integrability (on $\widehat{\mathcal{M}}_{D+1}$) fixes the scaling dimensions. From

$$\widehat{\nabla}\widehat{\mathbf{R}}^{\mathbf{p+1}} \approx \lambda N\Big(i\left([\widehat{e}_{(p+1)}, \Delta_{[p]}] - \widehat{e}_{(p+1)}\right)\widehat{\mathbf{X}}^{\mathbf{p}}$$

$$+ \sum_{p \geqslant p'+1}\left((\Delta_{[p]} + 1 + p)(\widehat{\sigma}_0^-)_{\mathbf{p'}}^{\mathbf{p+1}} - (\widehat{\sigma}_0^-)_{\mathbf{p'}}^{\mathbf{p+1}}(\Delta_{[p']} + p')\right)\widehat{\mathbf{X}}^{\mathbf{p'}}\Big) \qquad (3.11)$$

it follows that $\widehat{\mathbf{R}} \approx 0$ is integrable iff

$$\Delta_{[p]} = \Delta_{[p]}^f := \widehat{N}_{p+1}^{p+1} + f_{[p]}(\{\widehat{N}_i^i\}_{i=1,i\neq p+1}^{\nu}), \qquad (3.12)$$

$$\widehat{\mathbb{P}}(p+1, p'+1)\Big(t_{-1}(p'+1)\cdots t_{-1}(p)f_{[p]} + p - f_{[p']} - p'\Big) = 0. \qquad (3.13)$$

The last relation determines $f_{[p]}$ recursively in terms of a single function f,

$$f_{[p]} = -p + f\left(\widehat{N}_1^1 + 1, \ldots, \widehat{N}_p^p + 1, \widehat{N}_{p+2}^{p+2}, \ldots, \widehat{N}_\nu^\nu\right) \Rightarrow f_{[0]} = f(\widehat{N}_2^2, \widehat{N}_3^3, \ldots), \quad (3.14)$$

where the eigenvalues of $f_{[0]}$ are directly related to the lowest energy e_0 of the $\mathfrak{so}(2, D-1)$ lowest-weight space carried by the constrained system (see (3.52) below). The above form of $\Delta_{[p]}^f$ also implies that $(p \geqslant 1)$

$$\left(\widehat{\nabla} + \lambda N(\Delta_{[p]}^f + 1) - i\widehat{e}_{(p+1)}\right)\widehat{\mathbf{Z}}^{\mathbf{p}} + \sum_{p \geqslant p'+1}(\widehat{\sigma}_0^-)_{\mathbf{p'}}^{\mathbf{p+1}}\widehat{\mathbf{Z}}^{\mathbf{p'}} \approx 0. \qquad (3.15)$$

Finally, one has $(p \geqslant 1)$

$$\nabla\widehat{\mathbf{S}}^{\mathbf{p}} \approx \lambda N\Bigg[i\left([\widehat{e}_{(p+1)}, \Upsilon_{[p]}] - \widehat{e}_{(p+1)}\right)\widehat{\mathbf{Y}}^{\mathbf{p-1}} + (\Upsilon_{[p]} - \Delta_{[p]}^f - 1)\widehat{\mathbf{Z}}^{\mathbf{p}}$$

$$+ \sum_{p \geqslant p'+1}(-1)^{p-p'}\left((\Upsilon_{[p]} + 1 + p)(\widehat{\sigma}_0^-)_{\mathbf{p'}}^{\mathbf{p+1}} - (\widehat{\sigma}_0^-)_{\mathbf{p'}}^{\mathbf{p+1}}(\Upsilon_{[p']} + p')\right)\widehat{\mathbf{Y}}^{\mathbf{p'-1}}\Bigg], \qquad (3.16)$$

and hence $\widehat{\mathbf{S}} \approx 0$ is integrable iff

$$\Upsilon_{[p]} = \Delta_{[p]}^f + 1, \qquad (3.17)$$

as one may also deduce from dimensional analysis based on (3.5) and $N = dL$.

In summary, after radial reduction and constraining the radial derivatives we have

$$\widehat{\mathbf{R}}^{\mathbf{p+1}} := \left(\widehat{\nabla} + \lambda N\Delta_{[p]}^f - i\widehat{e}_{(p+1)}\right)\widehat{\mathbf{X}}^{\mathbf{p}} + \sum_{p \geqslant p'+1}(\widehat{\sigma}_0^-)_{\mathbf{p'}}^{\mathbf{p+1}}\widehat{\mathbf{X}}^{\mathbf{p'}} \approx 0, \qquad (3.18)$$

$$\widehat{\mathbf{S}}^{\mathbf{p}} := \left(\widehat{\nabla} + \lambda N(\Delta_{[p]}^f + 1) - i\widehat{e}_{(p+1)}\right)\widehat{\mathbf{Y}}^{\mathbf{p-1}}$$

$$+ \widehat{\mathbf{Z}}^{\mathbf{p}} + \sum_{p \geqslant p'+1}(-1)^{p-p'}(\widehat{\sigma}_0^-)_{\mathbf{p'}}^{\mathbf{p+1}}\widehat{\mathbf{Y}}^{\mathbf{p'-1}} \approx 0 \qquad (3.19)$$

where

$$\widehat{\mathbf{Z}}^{\mathbf{p}} := (\lambda \Delta_{[p]}^f + i\,\widehat{\xi}_{(p+1)})\widehat{\mathbf{X}}^{\mathbf{p}}$$
$$+i \sum_{p \geqslant p'+1} (p - p' + 1)\widehat{\xi}_{(p'+1)}\widehat{e}_{(p'+2)} \cdots \widehat{e}_{(p+1)}\widehat{\mathbb{P}}(p+1, p'+1)\widehat{\mathbf{X}}^{\mathbf{p}'}, \qquad (3.20)$$

which we note obeys (3.15). We denote the resulting module

$$\mathfrak{R}_f := \mathfrak{R}_{f,\perp} \oplus \mathfrak{R}_{f,\parallel}\,, \qquad (3.21)$$

where $\mathfrak{R}_{f,\perp} \ni \widehat{\mathbf{X}}$ and $\mathfrak{R}_{f,\parallel} \ni \widehat{\mathbf{Y}}$. The variables $(\widehat{\mathbf{Z}}(\widehat{\mathbf{X}}), \widehat{\mathbf{Y}})$ coordinatize a massively contractible cycle $\mathfrak{S}_f \subset \mathfrak{R}_f$ for all values of f.

3.2.2 Initial comments on criticality/reducibility

We recall from section I.4.4.4 that the potential submodule $\widetilde{\mathfrak{R}}$ of an unfolded module $\mathfrak{R}$ with Weyl zero-form module $\mathfrak{C}^{\mathbf{0}}$ is the maximal chain $\widetilde{\mathfrak{R}} := \widetilde{\mathfrak{R}}^{\mathbf{p}} \sqsupset \cdots \sqsupset \widetilde{\mathfrak{R}}^{\mathbf{p}'} \subset \mathfrak{R}$ with $p > 0$ whose elements cannot be set to zero for non-trivial Weyl zero-forms. Thus $\mathfrak{R} = \mathfrak{R}' \oplus \mathfrak{S}$ where $\mathfrak{R}' = \widetilde{\mathfrak{R}} \sqsupset \mathfrak{C}^{\mathbf{0}}$ and $\mathfrak{S}$ is massively contractible (cf. the example of massive spin-1 in flat spacetime discussed in section I.4.4.3).

For generic values of f, the map $(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \to (\widehat{\mathbf{X}}^{\mathbf{0}}, \widehat{\mathbf{Z}}(\widehat{\mathbf{X}}), \widehat{\mathbf{Y}})$ is an invertible (triangular) change of coordinates, i.e.

$$\text{generic } f : \; \mathfrak{R}_f \big|_{\mathfrak{g}_\lambda} \; = \; \mathfrak{S}_f \oplus \mathfrak{R}_{f,\perp}^{\mathbf{0}}\,, \qquad (3.22)$$

where $\mathfrak{C}_f^{\mathbf{0}} := \mathfrak{R}_{f,\perp}^{\mathbf{0}}$ is a massive Weyl zero-form module coordinatized by $\widehat{\mathbf{X}}^{\mathbf{0}}$, and $\mathfrak{S}_f$ is a massively contractible cycle coordinatized by $\{\widehat{\mathbf{Z}}^{\mathbf{p}}, \widehat{\mathbf{Y}}^{\mathbf{p}-\mathbf{1}}\}_{p>0}$. From (3.20) it follows that non-trivial potential modules arise iff f assumes critical values $\widetilde{f}$ such that

$$\text{non-trivial } \widetilde{\mathfrak{R}}_{\widetilde{f}} \Leftrightarrow \text{Ker}(\lambda\Delta_{[h_1]}^f + i\xi_{(h_1+1)}) \, \cap \, \mathfrak{R}_{\widetilde{f},\perp}^{\mathbf{h_1}} \; \neq \; \emptyset \qquad (3.23)$$

and the elements of $\text{Ker}(\lambda\Delta_{[h_1]}^f + i\xi_{(h_1+1)})$ are directly sourced by Weyl zero-forms (i.e., if they have maximal grade $\alpha = -1$).

On the other hand, as discussed in section I.4.3.4, $\mathfrak{C}_f^{\mathbf{0}}$ becomes reducible for the critical values $f_{I,N}^{\pm}$ of f corresponding to the critical masses $\overline{M}_{I,N}^2$ where primary Bianchi identities arise, and where $f^{\pm}$ refers to the two solutions of the characteristic equation (see (3.36) below). A subset of these, that we denote by $f_I^{\pm}$, correspond to critically massless fields with critical masses $\overline{M}_I^2$ defined in item (iii) of I.4.3.4 for which the system carries massless representations with a singular vector at the first level reached from the Ith block of the spin.

As we shall see, interestingly enough, there is only one critically massless $\widetilde{f}$, and it is given by

$$\widetilde{f} = f_1^-\,, \qquad (3.24)$$

and $\widetilde{\mathfrak{R}}_{f_1^-}$ consists of the unitary ASV potential.[3]

---

[3]At non-unitary critical values, namely $f_{p_I}^-$ for $p_I = \sum_{J=1}^I h_J$ with $I > 1$, the potential module of $\mathfrak{R}_{f_{p_I}^-}$ (as defined in section I.4.4.4) vanishes. It is still possible, however, to define a non-unitary ASV-like potential by partially gauge-fixing the massively contractible cycle. We thank E. Skvortsov for illuminating discussions on this point.

The fact that in AdS, differently from the flat-space case, the only two modules that can be glued together are a h_1-form module and the infinite-dimensional Weyl zero-form module is a direct consequence of Weyl's complete reducibility theorem, which forbids indecomposable finite-dimensional modules for a semi-simple Lie algebra (see also the comments in section I.3.4).

3.3 Radially reduced Weyl zero-form

3.3.1 Twisted-adjoint module and mass formula

The radially reduced Weyl zero-form obeys

$$\widehat{\mathbf{R}}^{\mathbf{1}} := \left(\widehat{\nabla} + \lambda N \Delta^f_{[0]} - i\,\widehat{e}_{(1)}\right) \widehat{\mathbf{X}}^{\mathbf{0}} \approx 0\,, \qquad \left(\mathcal{L}_\xi + \lambda \Delta^f_{[0]}\right) \widehat{\mathbf{X}}^{\mathbf{0}} \approx 0 \qquad (3.25)$$

in $D+1$ dimensions. The pull-back of the latter to AdS_D leaves with radius $L = \lambda^{-1}$ can be obtained using

$$\mathcal{L}_\xi \widehat{\mathbf{X}}^{\mathbf{0}} = i_\xi d\widehat{\mathbf{X}}^{\mathbf{0}} = i_\xi \widehat{\nabla}\widehat{\mathbf{X}}^{\mathbf{0}} \approx i\,\widehat{\xi}_{(1)}\widehat{\mathbf{X}}^{\mathbf{0}}\,, \qquad (3.26)$$

and that of $\widehat{\mathbf{R}}^{\mathbf{1}} \approx 0$ can be computed using $i^*_L \widehat{e}_{(i)} = e^a \widehat{\beta}_{a,(i)}$ and $i^*_L \widehat{\nabla} = \nabla - i\,\lambda\,e^a \widehat{\xi}^B \widehat{M}_{Ba}$ with $\nabla := d - \frac{i}{2}\omega^{ab}\widehat{M}_{ab}$, where $\widehat{M}_{AB}$ and $\widehat{M}_{ab}$ act canonically on $\widehat{\mathfrak{m}}$-types and their $\mathfrak{m}$-subtypes. Thus, at fixed λ one has

$$\mathbf{R}^{\mathbf{1}} := [\nabla - i\,e^a \rho(P_a)]\mathbf{X}^{\mathbf{0}} \approx 0\,, \qquad \rho(P_a) = \lambda\widehat{\xi}^B \widehat{M}_{Ba} + \widehat{\beta}_{a,(1)}\,, \qquad (3.27)$$

$$(\lambda \Delta^f_{[0]} + i\,\widehat{\xi}_{(1)})\widehat{\mathbf{X}}^{\mathbf{0}} \approx 0\,, \qquad \Delta^f_{[0]} = \widehat{N}^1_1 + f_{[0]}(\widehat{N}^2_2, \ldots, \widehat{N}^\nu_\nu)\,. \qquad (3.28)$$

Let us restrict $\widehat{\mathbf{X}}^{\mathbf{0}}$ to an irreducible twisted-adjoint $\mathfrak{iso}(2, D-1)$-module

$$\widehat{\mathcal{T}}(\Lambda=0; \overline{M}^2=0; \widehat{\overline{\Theta}})|_{\widehat{\mathfrak{m}}} = \bigoplus_{\alpha=0}^\infty \widehat{\Theta}_{[0];\alpha}\,, \quad \widehat{\Theta}_{[0];\alpha} = \left([s_1 + \alpha; 1], [s_1; h_1], \ldots, [s_B; h_B]\right), \, (3.29)$$

where the $\widehat{\mathfrak{m}}$-types are realized in $\mathcal{S}_{D+1}$ and descend from the smallest $\widehat{\mathfrak{m}}$-type

$$\widehat{\overline{\Theta}} := \widehat{\Theta}_{[0];0} = \left([s_1; h_1 + 1], [s_2; h_2], \ldots, [s_B; h_B]\right)\,, \qquad (3.30)$$

corresponding to the primary Weyl tensor of a tensor gauge field $\widehat{\varphi}(\Lambda = 0; M^2 = 0; \widehat{\overline{\Theta}})$ in $\mathbb{R}^{2,D-1}$ sitting in the $\widehat{\mathfrak{m}}$-type with shape $\widehat{\Theta} = \left([s_1; h_1], \ldots, [s_B; h_B]\right)$.

The constraint (3.25) yields a $\mathfrak{g}_\lambda$-module

$$\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}}) := \left\{\widehat{\mathbf{C}} \in \widehat{\mathcal{T}}(\Lambda=0; \overline{M}^2=0; \widehat{\overline{\Theta}}) : \left(\lambda\Delta^f_{[0]} + i\widehat{\xi}_{(1)}\right)\widehat{\mathbf{C}} \approx 0\right\}\,, \qquad (3.31)$$

that is irreducible for generic values of f and reducible with an indecomposable structure for critical values of f, determined by the value of

$$C_2\left[\mathfrak{g}_\lambda | \widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})\right] \equiv \left(C_2[\mathfrak{m}] - L^2\rho(P^2)\right)\big|_{\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})}\,, \qquad (3.32)$$

as discussed in section I.4.3.4 and below. The operator

$$-\rho(P^2) = -\left(\lambda^2\widehat{\xi}^B\widehat{\xi}^C\widehat{M}_B{}^a\widehat{M}_{Ca} + \lambda\widehat{\xi}^B\left\{\widehat{M}_B{}^a, \widehat{\beta}_{a,(1)}\right\} + \widehat{\beta}^a_{(1)}\widehat{\beta}_{a,(1)}\right) \tag{3.33}$$

$$= \lambda^2\left(\frac{1}{2}\widehat{M}^{AB}\widehat{M}_{AB} - \frac{1}{2}\widehat{M}^{ab}\widehat{M}_{ab}\right) - \lambda\,\widehat{\xi}^B\left\{\widehat{M}_B{}^A, \widehat{\beta}_{A,(1)}\right\} - \widehat{\beta}^A_{(1)}\widehat{\beta}_{A,(1)} - (\widehat{\xi}_{(1)})^2. \tag{3.34}$$

Using the relations (3.2), it follows that $-\rho(P^2) = \lambda^2(C_2[\widehat{\mathfrak{m}}] - C_2[\mathfrak{m}]) + i\,\lambda(2\widehat{N}_1^1 + D)\widehat{\xi}_{(1)} - (\widehat{\xi}_{(1)})^2$ where $C_2[\widehat{\mathfrak{m}}] := \frac{1}{2}\widehat{M}^{AB}\widehat{M}_{AB}$ and $C_2[\mathfrak{m}] := \frac{1}{2}\widehat{M}^{ab}\widehat{M}_{ab}$ are invariants for the action of $\widehat{\mathfrak{m}}$ and $\mathfrak{m}$ on $\widehat{\mathfrak{m}}$-types and their $\mathfrak{m}$-subtypes in $\widehat{\mathcal{S}}_{D+1}$. Further simplifications follow from

$$i\lambda(2\widehat{N}_1^1 + D)\widehat{\xi}_{(1)} \approx -\lambda^2(2\widehat{N}_1^1 + D)\Delta^f_{[0]}, \quad -(\widehat{\xi}_{(1)})^2 \approx \lambda^2(\Delta^f_{[0]} + 1)\Delta^f_{[0]}, \tag{3.35}$$

that hold in $\widehat{\mathcal{S}}(\Lambda; f; \overline{\widehat{\Theta}})$. Hence $-L^2\rho(P^2)\big|_{\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})} = C_2[\widehat{\mathfrak{m}}] - C_2[\mathfrak{m}] - (\widehat{N}_1^1 + f_{[0]})(\widehat{N}_1^1 + D - 1 - f_{[0]})$, and (3.32) takes the simplified form

$$C_2\left[\mathfrak{g}_\lambda|\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})\right] = \left(C_2[\widehat{\mathfrak{m}}] - (\widehat{N}_1^1 + f_{[0]})(\widehat{N}_1^1 + D - 1 - f_{[0]})\right)\Big|_{\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})}. \tag{3.36}$$

It follows that any given value μ of $C_2\left[\mathfrak{g}_\lambda|\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})\right]$ corresponds to two mass operators $f^\pm_{[0],\mu}$ given by

$$f^\pm_{[0],\mu} := \epsilon_0 + 1 \pm \sqrt{(\widehat{N}_1^1 + \epsilon_0 + 1)^2 + \mu - C_2[\widehat{\mathfrak{m}}]} \equiv f^\pm_\mu(\widehat{N}_2^2, \widehat{N}_3^3, \dots), \tag{3.37}$$

where the last identity can be seen by expanding the Casimir $C_2[\widehat{\mathfrak{m}}]$ — which is a nontrivial and non-constant operator in the module $\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})$. From (3.29) it can be seen that

$$f^\pm_{[0],\mu}|\widehat{\Theta}_{[0];\alpha}\rangle_{D+1} \equiv f^\pm_\mu(\Theta)|\widehat{\Theta}_{[0];\alpha}\rangle_{D+1} \qquad \forall \quad \alpha \tag{3.38}$$

where the α-independent massive parameter $f^\pm_\mu(\Theta)$ is given by

$$f^\pm_\mu(\Theta) := f^\pm_\mu(\underbrace{s_1, \dots, s_1}_{h_1 \text{ entries}}, \dots, \underbrace{s_B, \dots, s_B}_{h_B \text{ entries}}, 0, \dots). \tag{3.39}$$

Decomposing $\widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}})$ under $\mathfrak{m}$ the resulting smallest $\mathfrak{m}$-type is given by

$$\Theta = \left([s_1; h_1], \dots, [s_B; h_B]\right), \tag{3.40}$$

This shape is represented in the Schur module $\widehat{\mathcal{S}}_{D+1}$ by the state

$$|\Theta\rangle_{D+1} = \prod_{J=1}^B (\widehat{\xi}_{(p_J+1)})^{s_{J,J+1}}|\overline{\widehat{\Theta}}\rangle_{D+1} \tag{3.41}$$

belonging to the subspace $\mathcal{S}_D \subset \widehat{\mathcal{S}}_{D+1}$. The action of $\mathfrak{g}_\lambda$ on this state generates a $\mathfrak{g}_\lambda$-module. Removing the ideals (as we shall see, at most one non-trivial ideal arises) leaves an irreducible $\mathfrak{m}$-covariant $\mathfrak{g}_\lambda$-module with smallest type Θ, viz.

$$\mathcal{T}(\Lambda; \overline{M}^2; \Theta) := \bigoplus_{\alpha_r} \Theta_{\alpha_r}, \qquad |\Theta_{\alpha_r}| = |\Theta| + \alpha, \quad \Theta_0 = \Theta. \tag{3.42}$$

3.3.2 Proposition for indecomposability in the critical cases

We claim that, if $f_\mu^\pm$ denotes the two roots of the characteristic equation (3.36) for a fixed value $\mu = C_2[\mathfrak{g}_\lambda]$, then

$$\text{non-critical } f = f_\mu^\pm : \quad \widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}}) \;=\; \mathcal{T}(\Lambda; \overline{M}_f^2; \Theta), \tag{3.43}$$

$$\text{critical } f = f_\mu^\pm : \quad \widehat{\mathcal{S}}(\Lambda; f; \widehat{\overline{\Theta}}) \;=\; \begin{cases} \mathcal{T}(\Lambda; \overline{M}_f^2; \Theta) \Subset \mathcal{T}(\Lambda; \overline{M}_f'^2; \overline{\Theta}_f) \text{ for } f = f_\mu^- \\ \mathcal{T}(\Lambda; \overline{M}_f^2; \Theta) \Supset \mathcal{T}(\Lambda; \overline{M}_f'^2; \overline{\Theta}_f) \text{ for } f = f_\mu^+ \end{cases} \tag{3.44}$$

where

(i) for non-critical $f_\mu^\pm$, $\mathcal{T}(\Lambda; \overline{M}_{f_\mu^+}^2; \Theta) \cong \mathcal{T}(\Lambda; \overline{M}_{f_\mu^-}^2; \Theta)$ (that is, $\overline{M}_{f_\mu^+}^2 \equiv \overline{M}_{f_\mu^-}^2$) is a generically massive twisted-adjoint (irreducible) $\mathfrak{g}_\lambda$-module (see section I.4.3.1); and

(ii) at critical $f_\mu^\pm$, two dual indecomposable structures arise: The representation matrices are transposed upon exchanging f_μ^- with f_μ^+.

In section 3.4 we prove (3.43) in general.

In section 3.5 we then prove a part of the claim (3.44), namely that in the critically massless cases (see item (iii) I.4.3.4) it follows that

$$\text{critically massless } f = f_I^- : \quad \mathcal{T}(\Lambda; \overline{M}_{f_I^-}'^2; \overline{\Theta}_{f_I^-}) = \mathcal{T}(\Lambda; \overline{M}_I^2; \overline{\Theta}_I), \tag{3.45}$$

$$\mathcal{T}(\Lambda; \overline{M}_{f_I^-}^2; \Theta) = \mathcal{T}(\Lambda; \overline{M}_{I, S_{I,I+1}}^2; \Theta), \tag{3.46}$$

where

$$\overline{\Theta}_I := \Big([s_1; h_1], \ldots, [s_{I-1}; h_{I-1}], [s_I; h_I + 1], [s_{I+1}; h_{I+1} - 1], [s_{I+2}; h_{I+2}], \ldots, [s_B; h_B]\Big). \tag{3.47}$$

We identify the above two modules, respectively, as the twisted-adjoint representations of the primary Weyl tensors C_{φ_I} and C_{χ_I} of Metsaev's critically massless gauge fields φ_I, and of the corresponding Stückelberg fields χ_I associated with massive gauge symmetries in the Ith block i.e.

$$C_{\varphi_I}(\Lambda; \overline{M}_I^2; \overline{\Theta}_I) \;\overset{\text{integrate}}{\rightsquigarrow}\; \varphi_I(\Lambda; M_I^2; \Theta),$$
$$C_{\chi_I}(\Lambda; \overline{M}_{I, S_{I,I+1}}^2; \Theta) \;\overset{\text{integrate}}{\rightsquigarrow}\; \chi_I(\Lambda; M_I'^2; \Theta_I'), \tag{3.48}$$

where Θ_I' is obtained by deleting one cell from the Ith block of Θ, viz.

$$\Theta_I' = \Big([s_1; h_1], \ldots, [s_{I-1}; h_{I-1}], [s_I; h_I - 1], [s_I - 1; 1], [s_{I+1}; h_{I+1}], \ldots, \ldots, [s_B; h_B]\Big). \tag{3.49}$$

These Stückelberg fields $\chi_I(\Lambda; M_I'^2; \Theta_I')$ are partially massless,[4] in accordance with our general definition in item (iv) of section I.4.3.4, whenever any block $I = 2, \ldots, B$ of Θ is of height one — while the case $h_1 = 1 = I$ instead gives cut twisted-adjoint modules, as

---

[4]Actually, setting $I = 2 = B$, one obtains shapes $\widehat{\overline{\Theta}} = (s, s, t)$ corresponding to a partially massless Stückelberg field $\chi(s, t-1)$ having gauge invariance $\delta\chi(s, t-1) = (\overline{\nabla}^{(1)})^{s-t+1}\epsilon(t-1, t-1)$. This field reduces to a non-generic partially massless symmetric tensor of [41] iff $t = 1$.

defined in item (ii) of section I.4.3.4. In the latter case, these cut modules actually arise from factoring out a tensorial $\mathfrak{g}_\lambda$-module (see item (i) of section I.4.3.4) from the Weyl zero-form module generated from a primary Weyl tensor of the same shape as $\chi_I(\Theta_I')$.

The fact that $\chi_I(\Lambda; M_I'^2; \Theta_I')$ can be factored out is a manifestation of the fact that there is enhancement of gauge symmetry in the Ith block: The radially reduced $(D+1)$-dimensional gauge field $\widehat{\varphi}(\widehat{\Theta})$ (with constrained radial derivatives) decomposes into

$$\widehat{\varphi}(\widehat{\Theta}) \;\to\; \varphi(\Theta) \cup \left\{\chi(\Theta_I')\right\}_{I=1}^B \cup \left\{\chi(\Theta_{I,J}'')\right\}_{I,J=1}^B \cup \cdots,$$

where Θ_I' is obtained by deleting one cell from the Ith block of Θ, $\Theta_{I,J}'' \equiv \Theta_{J,I}''$ is obtained by deleting one cell from the last row of the Jth block of Θ_I', and so on. For generic mass all Stückelberg fields are "eaten" by the massive field $\varphi(\Lambda; M^2; \Theta)$. To examine the critical limit $\zeta_I^2 := (M^2 - M_I^2) \to 0$ (fixed I) one may arrange the reduced field content as follows:

$$\widehat{\varphi}(\widehat{\Theta}) \;\to\; \underbrace{\left\{\varphi(\Theta) \cup \left\{\chi(\Theta_J')\right\}_{J=1;J\neq I}^B \cup \left\{\chi(\Theta_{J,K}'')\right\}_{J,K=1;J,K\neq I}^B \cup \cdots,\right\}}_{=:\widehat{\varphi}_I} \tag{3.50}$$

$$\cup \;\; \underbrace{\left\{\chi(\Theta_I') \cup \left\{\chi(\Theta_{I,J}'')\right\}_{J=1}^B \cup \cdots\right\}}_{=:\widehat{\chi}_I}. \tag{3.51}$$

In the limit $\zeta_I^2 \to 0$ there is enhancement of gauge symmetry in the Ith block which means that the $\widehat{\varphi}_I$ system decouples from $\widehat{\chi}_I$ that becomes an independent — generically partially massless — field system. One may remove $\widehat{\chi}_I$ from the equations of motion/action by fixing the gauge $\widehat{\chi}_I \overset{!}{=} 0$ for $\zeta_I^2 \neq 0$ (which involves division by ζ_I^2) and then send ζ_I^2 to zero. The equations of motion/action remain smooth in this limit though the number of degrees of freedom change.

3.4 The generically massive case

Let us show that the generic Weyl zero-form module $\mathfrak{C}^0(\Lambda; \overline{M}^2; \Theta)$ carries the massive representations $\mathfrak{D}(e_0; \Theta)$ with

$$e_0 = \begin{cases} f(\Theta) \\ D - 1 - f(\Theta) \end{cases}. \tag{3.52}$$

3.4.1 Harmonic expansion

To this end we first construct the harmonic map [42][5]

$$\mathcal{S}_\mathcal{T} : \;\mathfrak{D}_{D-1}^+(e_0; \Theta) \to \mathfrak{C}^0(\Lambda; \overline{M}^2; \Theta), \tag{3.53}$$

where $\mathfrak{D}_{D-1}^+(e_0; \Theta) := [\mathfrak{D}^+(e_0; \Theta) \otimes \mathcal{S}_{D-1}]_{\text{diag}}$ is the subspace of $\mathfrak{D}^+(e_0; \Theta) \otimes \mathcal{S}_{D-1}$ consisting of states that are invariant under $\mathfrak{s}_{\text{diag}} = (\mathfrak{s}^+ \oplus \widehat{\mathfrak{s}})_{\text{diag}}$ generated by

$$M_{rs}^{\text{diag}} = M_{rs} + \widehat{M}_{rs}, \tag{3.54}$$

---

[5]The map extends to real Weyl tensors in $(\mathcal{S}_\mathcal{T}^- \mathfrak{D}_{D-1}^+) \oplus (\mathcal{S}_\mathcal{T}^- \mathfrak{D}_{D-1}^-)$ where $\mathfrak{D}^\pm(\pm e_0; \Theta)$ are lowest-weight (+) and highest-weight (−) spaces.

where M_{rs} act in $\mathfrak{D}^+(e_0;\Theta)$ and $\widehat{M}_{rs}$ act in $\mathcal{S}_{D-1}$. The diagonal states are

$$|e_0 + m + n; \theta\rangle^+_{D-1} := L^{+,(j_1)} \cdots L^{+,(j_m)} L^+_{(k_1)} \cdots L^+_{(k_n)} |e_0; \Theta\rangle^+_{D-1} , \qquad (3.55)$$

where $L^{+,(j)} := \bar{\beta}^{r,(j)} L^+_r$ modulo traces terms and $L^+_{(j)} := \beta^r_{(j)} L^+_r$ with $\overrightarrow{w}(\theta) = \overrightarrow{w}(\Theta) + \overrightarrow{w}$ such that $w_j = \sum_{l=1}^m \delta_{j,j_l} - \sum_{l=1}^n \delta_{j,k_l}$, $j_l \geqslant j_{l+1}$, $k_l \leqslant k_{l+1}$. The diagonal ground state obeys

$$L^-_r |e_0; \Theta\rangle^+_{D-1} = 0, \quad (E - e_0) |e_0; \Theta\rangle^+_{D-1} = 0 . \qquad (3.56)$$

For generic e_0 there are no singular vectors.

Decomposing under $\mathfrak{s}^+$ yields

$$\mathfrak{D}(e_0;\Theta)^+_{D-1}\big|_{\mathfrak{s}^+} = \bigoplus_{\theta \in \mathcal{S}(e_0;\Theta)} \overset{\infty}{\underset{n=0}{\bigoplus}} \mathbb{C} \otimes \left(x^n |e(\theta); \theta\rangle^+_{D-1} \right) , \quad x := \delta^{rs} L^+_r L^+_s , \quad (3.57)$$

where $\mathcal{S}(e_0;\Theta)$ is the set of $\mathfrak{s}^+$-types arising in $\mathfrak{D}^+(e_0;\Theta)$. This set contains a unique minimal $\mathfrak{s}^+$-type θ'_0. The lowest-spin state $|e'_0; \theta'_0\rangle^+_{D-1}$ is defined to be the state of minimal $\mathfrak{s}^+$-type that minimizes the energy (see figure 2). By its definition this state obeys

$$L^+_{(j)} |e'_0; \theta'_0\rangle^+_{D-1} = 0 \qquad \forall j . \qquad (3.58)$$

Under the assumption that there are no singular vectors, it follows that $(p_J = \sum_{K=1}^J h_K)$

$$|e'_0; \theta'_0\rangle^+_{D-1} = \prod_{J=1}^B (L^+_{(p_J)})^{s_{J,J+1}} |e_0; \Theta\rangle^+_{D-1}, \quad e'_0 = e_0 + s_1 . \qquad (3.59)$$

To show (3.53) it suffices to map $\mathfrak{D}^+_{D-1}(e_0;\Theta)$ to the primary (massive) Weyl tensor $C(\Theta) \in \mathfrak{C}^{\mathbf{0}}(\Lambda; \overline{M}^2; \Theta)$. This tensor belongs to $\mathcal{S}_D$ due to (3.41) and decomposes under $\widehat{\mathfrak{s}}$ as follows:

$$C(\Theta)\big|_{\widehat{\mathfrak{s}}} = \sum_{\theta \in \Theta|_{\widehat{\mathfrak{s}}}} \prod_{J=1}^B (\bar{\beta}_0^{(p_J+1)})^{n_J(\theta|\Theta)} C(\theta|\Theta) , \quad C(\theta|\Theta) \in \Omega^{\mathbf{0}}(U) \otimes \theta , \qquad (3.60)$$

where $n_J(\theta|\Theta)$ is the number of boxes which are removed from the Jth block of Θ in order to obtain θ. It follows that the smallest $\widehat{\mathfrak{s}}$-type of $C(\Theta)$, i.e. its most electric component, is given by $C(\theta'_0|\Theta)$. Let us seek a harmonic expansion given by the Ansatz (cf. totally symmetric massless tensors [42])

$$C(\theta'_0|\Theta) = \sum_{(e,\theta)}{}^+ \langle C^*_{(e,\theta)} | L^+ | \theta'_0 | \Theta \rangle , \qquad (3.61)$$

$$|\theta'_0|\Theta\rangle := \psi_{\theta'_0|\Theta}(x) |e'_0; \theta'_0\rangle , \quad \psi_{\theta'_0|\Theta}(x) := \sum_{n=0}^{\infty} x^n \psi_{n;\theta'_0|\Theta} , \qquad (3.62)$$

where:

i) $^+\langle C^*_{(e,\theta)}| := C^+_{(e,\theta)} {}^+\langle e;\theta| \in [\mathfrak{D}^+(e_0;\Theta)]^*$ are states with fixed energy and spin;

ii) L^+ is a coset representative of AdS_D acting in $\mathfrak{D}^+(e_0;\Theta)$; and

iii) the embedding function $\psi_{\theta'_0|\Theta}(x)$ is determined by demanding $|\theta'_0|\Theta\rangle$ to be an $\mathfrak{s}$-type in Θ, i.e.

$$\bar{\beta}^{r,(1)} M_{0r}|\theta'_0|\Theta\rangle - \text{traces} = 0 \quad \text{where } M_{0r} = \tfrac{1}{2}(L^+_r + L^-_r)\,. \tag{3.63}$$

The latter condition amounts to that

$$M_{0\{r_1}\psi_{\theta'_0|\Theta}(x)|e'_0;\theta'_0\rangle_{r_1(s_1)\},...,r_{h_1}(s_1);...;t_1(s_B),...,t_{h_B}(s_B)} = 0\,. \tag{3.64}$$

Using the commutation relations (A.8) which yield the useful relation

$$[L^-_r, x^n] = 4nx^{n-1}(iL^+_s M_{rs} + L^+_r(E + n - \epsilon_0 - 1))\,,$$

the embedding condition can be rewritten as

$$L^+_{\{r_1} D_2 \psi_{\theta'_0|\Theta}(x)|e'_0;\theta_0\rangle_{r_1(s_1)\},...,r_{h_1}(s_1);...;t_1(s_B),...,t_{h_B}(s_B)} = 0\,, \tag{3.65}$$

where $\{\cdots\}$ denotes symmetric and traceless projection, and

$$D_2 := 4x\frac{d^2}{dx^2} + 4(e_0 - \epsilon_0)\frac{d}{dx} + 1\,. \tag{3.66}$$

It follows that there exists a regular embedding function given by the rescaled Bessel function

$$\psi_{\theta'_0|\Theta}(x) = (\sqrt{x})^{-\nu} J_\nu(\sqrt{x})\,, \qquad \nu = e_0 - \epsilon_0 - 1\,. \tag{3.67}$$

3.4.2 Characteristic equation

Finally, the values (3.52) of the lowest energy e_0 are determined by the characteristic equation

$$C_2\left[\mathfrak{g}_\lambda|\mathfrak{D}(e_0;\Theta)\right] = C_2[\mathfrak{g}_\lambda|\mathcal{T}(\Lambda;\overline{M}^2;\Theta)]\,, \tag{3.68}$$

where $C_2[\mathfrak{g}_\lambda|\mathfrak{D}(e_0;\Theta)] = e_0(e_0 - 2\epsilon_0 - 2) + C_2[\mathfrak{s}|\Theta]$ with $\mathfrak{s} = \mathfrak{so}(D-1)$, and the right-hand side is given by (3.36). Using the parametrization of $\widehat{\mathfrak{m}}$-types given in (3.29), one finds

$$C_2[\widehat{\mathfrak{m}}|\widehat{\Theta}_{\alpha_i}] = C_2[\mathfrak{s}|\Theta] + (s_1 + \alpha)(s_1 + \alpha + 2\epsilon_0 + 2)\,, \tag{3.69}$$

leading to the following form of the characteristic equation (3.68):

$$(e_0 - \epsilon_0 - 1)^2 = (\epsilon_0 + 1)^2 + (s_1 + \alpha)(s_1 + \alpha + 2\epsilon_0 + 2) - (s_1 + \alpha + f(\Theta))(s_1 + \alpha + 2\epsilon_0 + 2 - f(\Theta))$$
$$= (f(\Theta) - \epsilon_0 - 1)^2\,, \tag{3.70}$$

with the roots (3.52).
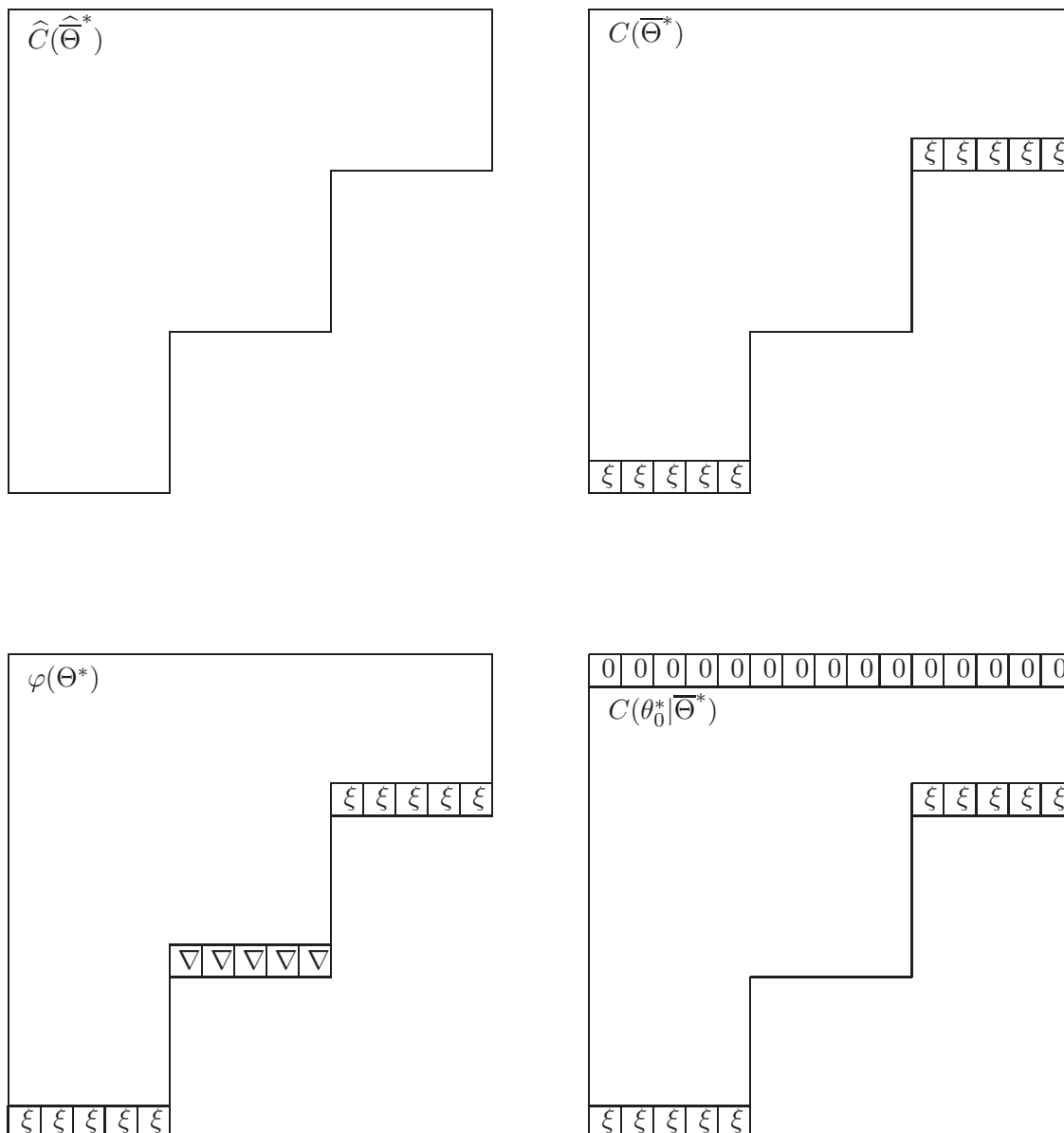
Figure 1. The four shapes associated with (1) the original strictly massless primary Weyl tensor in $\mathbb{R}^{2,D-1}$; (2) the reduced, critically massless primary Weyl tensor in AdS_D; (3) the corresponding critically massless gauge potential in AdS_D; and (4) the most electric component of (2).

3.5 The critically massless case

3.5.1 Proof of indecomposability for massless cases

Let us first show (3.45). To this end let us seek the critical values $f_{_I}$ of f for which the representation $\rho(P_a)$ in the $\mathfrak{g}_\lambda$-module $\widehat{\mathcal{S}}(\Lambda; f_{_I}; \overline{\widehat{\Theta}})$ defined in (3.31) becomes indecomposable with ideal

$$\mathcal{T}_\chi(\Lambda; \overline{M}^2_{I,S_{I,I+1}}; \Theta) := \mathrm{Im}(\widehat{\xi}_{p_I+1}) \cap \widehat{\mathcal{S}}(\Lambda; f; \overline{\widehat{\Theta}}) = \{\widehat{\xi}_{(p_I+1)}\widehat{\mathbf{C}} \text{ for } \widehat{\mathbf{C}} \in \widehat{\mathcal{S}}(\Lambda; f; \overline{\widehat{\Theta}})\}. \ (3.71)$$

where

$$p_I := \sum_{J=1}^{I} h_J .$$

Setting this ideal to zero amounts to constraining the Weyl zero-form module as follows:

$$
\begin{array}{l}
(p_I + 1)\text{-row projection} \\
(\text{freezing } (p_I + 1)\text{st row} \qquad : \widehat{\xi}_{(p_I+1)}\widehat{X}^{\mathbf{0}} \approx 0 . \\
\text{in the primary Weyl tensor})
\end{array}
\tag{3.72}
$$

Cartan integrability of the above constraint, which is equivalent to the ideal property of $\mathcal{T}_\chi(\Lambda; \overline{M}^2_{I,S_{I,I+1}}; \Theta)$, amounts to

$$\left(\lambda \widehat{e}_{(p_I+1)} + i[\widehat{\xi}_{(p_I+1)}, \widehat{e}_{(1)}] \right) \widehat{X}^{\mathbf{0}} \equiv 0 \qquad \text{modulo } \left(\lambda \Delta^f_{[0]} + i\widehat{\xi}_{(1)} \right) \widehat{X}^{\mathbf{0}} \approx 0, \tag{3.73}$$

where $\Delta^f_{[0]} := \widehat{N}^1_1 + f_{[0]}$ are the scaling dimensions appearing in the radial velocity constraints (3.10). We claim that this equation has the unique solution[6]

$$f = f_{p_I} := p_I + 1 - \widehat{N}^{p_I+1}_{p_I+1} \Rightarrow \Delta^{f_{p_I}}_{[p]} = \widehat{N}^{p+1}_{p+1} + \begin{cases} p_I + 1 - p - \widehat{N}^{p_I+1}_{p_I+1} & \text{for } p \leqslant p_I - 1, \\ p_I - p - \widehat{N}^{p_I}_{p_I} & \text{for } p \geqslant p_I \end{cases} \tag{3.74}$$

We have shown this for $\nu = p_I + 1 \geqslant 2$ (in which case $\widehat{\xi}_{(p_I+1)} = \widehat{\xi}_A \alpha^A_\nu$ that simplifies the calculations somewhat) and $\nu = p_I + 2 = 3$ using the explicit expression (2.55) for the cell operators. For fixed $\widehat{\overline{\Theta}}$ it follows that $\widehat{\mathcal{S}}(\Lambda; f_{p_I}; \widehat{\overline{\Theta}})$ contains the proper submodule $\mathcal{T}(\Lambda; \overline{M}^2_I; \overline{\Theta}_I)$ with primary type of shape $\overline{\Theta}_I$ given by (3.47) represented in $\widehat{\mathcal{S}}_{D+1}$ by

$$|\overline{\Theta}_I\rangle_{D+1} = \prod_{\substack{J=1 \\ J \neq I}}^{B} (\widehat{\xi}_{(p_J+1)})^{s_{J,J+1}} |\widehat{\overline{\Theta}}\rangle_{D+1},$$

$$\widehat{\xi}_{(p_J)}|\overline{\Theta}_I\rangle_{D+1} = 0 \, (J \neq I), \qquad \widehat{\xi}_{(p_I+1)}|\overline{\Theta}_I\rangle_{D+1} = 0, \tag{3.75}$$

which means that $|\overline{\Theta}_I\rangle_{D+1} \in \mathcal{S}_D \subset \widehat{\mathcal{S}}_{D+1}$. This embedding implies that $\rho(P_a)|\overline{\Theta}_I\rangle_{D+1}$ cannot be anti-symmetrized into the Ith block. It follows that the generalized Verma module $\mathcal{V}^*(\Lambda; \overline{M}^2_I; \overline{\Theta}_I)$ contains a singular vector corresponding to the primary Bianchi identity

$$\overline{\nabla}^{[s_{I+1}+1]} C(\Lambda; \overline{M}^2_I; \overline{\Theta}_I) = 0 . \tag{3.76}$$

Integration yields the gauge field $\varphi(\Lambda; M^2_I; \Theta)$ in AdS_D sitting in the same $\mathfrak{m}$-type Θ as the generically massive gauge field $\varphi(\Lambda; M^2; \Theta)$, given by (3.40). According to the nomenclature of section I.4.3.4 the field $\varphi(\Lambda; M^2_I; \Theta)$ is massless except if $I = B$ and $h_B = 1$ in which case it is partially massless.[7]

---

[6]One consequence of (3.74) is that $\widehat{X}^{\mathbf{1}a}$ and $\widehat{e}^a$ have the same scaling dimensions, viz. $(\mathcal{L}_\xi - \lambda)\widehat{X}^{\mathbf{1}a} = (\mathcal{L}_\xi - \lambda)\widehat{e}^a = 0$, so that the "graviton field" $\widehat{X}^{\mathbf{1}a}$ can consistently deform the background vielbein $\widehat{e}^a$ upon switching on interactions.

[7]If $B = 1 = h_B$, namely, only one block of height one (totally symmetric case), this reduces to the case first investigated in [43], later revisited in [41]. See also [44, 45].

Identifying $f_{p_I} \equiv f_{p_I}^-$ it follows that $C_2[\mathfrak{g}_\lambda]$ assumes the same value in $\widehat{\mathcal{S}}(\Lambda; f_{p_I}^+; \widehat{\overline{\Theta}})$ where

$$f_+^+ := D - 1 - f_-^{p_I} = \widehat{N}_{1+p_I}^{1+p_I} + D - p_I - 2 \equiv e_0^I \,. \tag{3.77}$$

Hence $\widehat{\mathcal{S}}(\Lambda; f_{p_I}^+; \widehat{\overline{\Theta}})$ must consist of the same twisted-adjoint representations as $\widehat{\mathcal{S}}(\Lambda; f_{p_I}^-; \widehat{\overline{\Theta}})$. But $\widehat{\mathcal{S}}(\Lambda; f_{p_I}^+; \widehat{\overline{\Theta}})$ does not contain $\mathcal{T}(\Lambda; \overline{M}_I^2; \overline{\Theta}_I)$ as an ideal. We claim that the indecomposable structure of $\widehat{\mathcal{S}}(\Lambda; f_{p_I}^+; \widehat{\overline{\Theta}})$ takes the form given in eq. (3.44) for $f = f^+$, in other words, the decomposition order is reversed with respect to that of $\widehat{\mathcal{S}}(\Lambda; f_{p_I}^-; \widehat{\overline{\Theta}})$. It appears to us that the reversed indecomposable structure cannot be characterized by means of any algebraic subsidiary condition involving $\widehat{\xi}^A$ contractions.

Referring to item (iv) in section I.4.3.4 it is plausible that the following generalization of (3.72):

$(p_I + 1)$-row projection $(k \geqslant 1)$
(reducing $k - 1$ cells in $(p_I + 1)$st row : $\quad (\widehat{\xi}_{(p_I+1)})^k \widehat{X}^0 \approx 0 \,,$ (3.78)
in the primary Weyl tensor)

which — as we already have shown — leads to mixed-symmetry massless fields if $k = 1$, will give rise to mixed-symmetry partially massless fields if $k \geqslant 2$, since the projection then creates a block of height one in the primary Weyl tensor. We leave this for future work.

3.5.2 Harmonic expansion via most electric primary Weyl tensor

In the critical limit the larger of the two characteristic energies in (3.52) becomes $e_0^I := s_I + D - 2 - p_I$ corresponding to the massless lowest-weight irrep $\mathfrak{D}^+(e_0^I; \Theta)$ with singular vector

$$L_{(p_I)}^+ |e_0^I; \Theta\rangle_{D-1}^+ \approx 0 \,, \tag{3.79}$$

presented here as a state in the doubled space $[\mathfrak{D}^+(e_0^I; \Theta) \otimes \mathcal{S}_{D-1}]_{\mathrm{diag}}$, using the notation of section 3.4. Let us show that this irrep is carried by $C(\Lambda; \overline{M}_I^2; \overline{\Theta}_I)$, i.e. that there exists a harmonic map

$$\mathcal{S}^{C_{\mathrm{electric}}} : \left[\mathfrak{D}^+(e_0^I; \Theta) \otimes \mathcal{S}_{D-1}\right]_{\mathrm{diag}} \to \mathfrak{C}^0(\Lambda; \overline{M}_I^2; \overline{\Theta}_I) \,, \tag{3.80}$$

with reference state given by the most electric component of the primary Weyl tensor.

To this end we note that the existence of the singular vector (3.79) implies that

$$\mathfrak{D}(e_0^I; \Theta)_{D-1}^+\big|_{\mathfrak{s}^+} = \bigoplus_{\theta \in \mathcal{S}} \bigoplus_{n=0}^{\infty} \mathbb{C} \otimes \left(x^n |e(\theta); \theta\rangle_{D-1}^+\right) \,, \quad x := \delta^{rs} L_r^+ L_s^+ \,, \tag{3.81}$$

where $\mathcal{S}(e_0^I; \Theta)$, the set of $\mathfrak{s}^+$-types arising in $\mathfrak{D}^+(e_0^I; \Theta)$, is smaller than in the massive case presented above since the operator $L_{(p_I)}^+$ annihilates $|e_0^I; \Theta\rangle_{D-1}^+$. In other words, the lowest-spin state $|e_0^{I\prime}; \theta_0^I\rangle_{D-1}^+$ is now given by

$$|e_0^{I\prime}; \theta_0^I\rangle_{D-1}^+ = \prod_{J=1, J\neq I}^{B} (L_{(p_J)}^+)^{s_{J,J+1}} |e_0^I; \Theta\rangle_{D-1}^+ \,, \quad e_0^{I\prime} = e_0^I + s_1 - s_I + s_{I+1} \,. \tag{3.82}$$

On the other hand, the primary Weyl tensor $C(\overline{\Theta}_I) \in \mathfrak{C}^{\mathbf{0}}(\Lambda; \overline{M}_I^2; \overline{\Theta}_I)$ belongs to $\mathcal{S}_D$ due to (3.75). Therefore, it decomposes under $\mathfrak{s}$ as follows:

$$C(\overline{\Theta}_I)\big|_{\mathfrak{s}} = \sum_{\overline{\theta} \in \overline{\Theta}_I|_{\mathfrak{s}}} (\bar{\beta}_0^{(p_I+1)})^{n_I} \prod_{J=1, J\neq I}^{B} (\bar{\beta}_0^{(p_J)})^{n_J} C(\overline{\theta}|\overline{\Theta}_I) , \quad C(\overline{\theta}|\overline{\Theta}_I) \in \Omega^0(U) \otimes \overline{\theta}. \quad (3.83)$$

We then seek a harmonic expansion for the most electric component, viz.

$$C(\theta_0^I|\overline{\Theta}_I) = \sum_{(e,\theta)}{}^+ \langle C^*_{(e,\theta)}|L^+|\theta_0^I|\overline{\Theta}_I\rangle , \qquad |\theta_0^I|\overline{\Theta}_I\rangle := \psi_{\theta_0^I|\overline{\Theta}_I}(x)|e_0^{I\prime}; \theta_0^I\rangle , \quad (3.84)$$

where the embedding function obeys

$$M_{0\{r_1} \psi_{\theta_0^I|\overline{\Theta}_I}(x)|e_0^{I\prime}; \theta_0^I\rangle_{r_1(s_1)\},\dots,r_{h_1}(s_1);\dots;t_1(s_B),\dots,t_{h_B}(s_B)} = 0 . \quad (3.85)$$

This implies the second-order differential equation $D_2 \psi_{\theta_0^I|\overline{\Theta}_I} = 0$ with

$$D_2 = 4x\frac{d^2}{dx^2} + 4(\epsilon_0 + 1 + s_{I+1} - p_I)\frac{d}{dx} + 1 , \quad (3.86)$$

leading to the regular embedding function

$$\psi_{\theta_0^I|\overline{\Theta}_I}(x) = (\sqrt{x})^{-\nu_I} J_{\nu_I}(\sqrt{x}), \qquad \nu_I = \epsilon_0 + s_{I+1} - p_I . \quad (3.87)$$

Thus $\mathfrak{D}(e_0^I; \Theta)$ is carried by $C(\overline{\Theta}_I)$, and hence by all elements of $\mathfrak{C}^{\mathbf{0}}(\Lambda; \overline{M}_I^2; \overline{\Theta}_I)$. $\square$

3.5.3 Harmonic expansion via most magnetic Weyl tensor and shadow

One can also show that there exists a harmonic map with reference state given by the most magnetic component of the primary Weyl tensor as follows:

$$\mathcal{S}^{C_{\mathrm{magn}}} : \big[\mathfrak{D}^+(\check{e}_0^I; \overline{\Theta}_I) \otimes \mathcal{S}_{D-1}\big]_{\mathrm{diag}} \rightarrow \mathfrak{C}^{\mathbf{0}}(\Lambda; \overline{M}_I^2; \overline{\Theta}_I) , \quad (3.88)$$

where the lowest-energy is given by

$$\check{e}_0^I = 1 + p_I - s_{I+1} . \quad (3.89)$$

This is a direct generalization of the special case $B = 1$, $h_1 = 1$ spelled out for composite massless fields in [42].

The critical limit of the smaller energy eigenvalue in (3.52) is given by $\widetilde{e}_0^I := D - 1 - e_0^I$. This energy corresponds to the shadow $\mathfrak{so}(2, D-1)$- module $\mathfrak{D}(\widetilde{e}_0^I; \Theta)$. This module has a different pattern of singular vectors. It has no singular vector with rank smaller than $|\Theta|$. Hence its lowest-spin state $|\widetilde{e}_0'; \theta_0'\rangle$ has the same $\mathfrak{s}^+$-spin θ_0' as in the generically massive case analyzed in section 3.4. It follows that there exists a harmonic map

$$\mathcal{S}^{\varphi} : \big[\mathfrak{D}^+(\widetilde{e}_0^I; \Theta) \otimes \mathcal{S}_{D-1}\big]_{\mathrm{diag}} \rightarrow \varphi(\Lambda; \overline{M}_I^2; \Theta) , \quad (3.90)$$

so that $\varphi(\Lambda; \overline{M}_I^2; \Theta)$ carries $\mathfrak{D}^+(\widetilde{e}_0^I; \Theta)$.
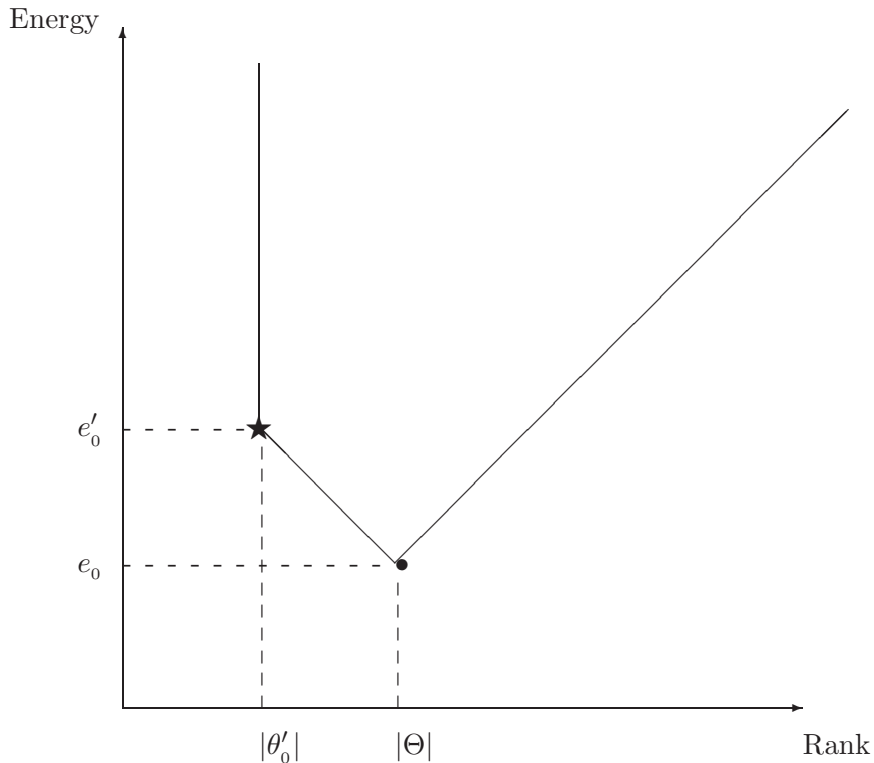
Figure 2. A lowest-weight module $\mathfrak{D}(e_0;\Theta)$ with its lowest-energy state $|e_0;\Theta\rangle$ and the lowest-spin state $|e_0';\theta_0'\rangle$ indicated by the $\bullet$ and $\bigstar$, respectively.

3.6 Unitarizable ASV potential

3.6.1 Occurrence of non-trivial potential module

Let us consider the subsector $\mathfrak{R}_f(\Lambda;\Theta) \subset \mathfrak{R}_f$ obtained by constrained radial reduction of the gauge field $\widehat{\varphi}(\Lambda=0;\widehat{\Theta})$ with $\widehat{\Theta} = ([s_1;h_1],\ldots,[s_B;h_B])$ so that $f_{[0]}$ can be replaced by its eigenvalue $f(\Theta)$ and let us denote its potential module by $\widetilde{\mathfrak{R}}_f(\Lambda;\Theta)$. For generic f, $\mathfrak{R}_{f,\perp}^{\mathbf{h_1}}$ belongs to $\mathfrak{S}_f(\Theta)$, that in its turn implies that all p-forms with $p > 0$ belongs to $\mathfrak{S}_f(\Theta)$. This is so even in case $\mathrm{Ker}(\lambda\Delta_{[p_I]}^f + i\xi_{(p_I+1)}) \cap \mathfrak{R}_{f,\perp}^{\mathbf{p_I}}$ is non-empty for some $I > 1$, because higher-degree potentials are not sourced directly by the Weyl zero-form. Thus, $\mathfrak{R}_f(\Lambda;\Theta)$ contains a non-trivial potential (in the sense explained in section I.4.4.4) iff $\mathrm{Ker}(\lambda\Delta_{[h_1]}^f + i\xi_{(h_1+1)}) \cap \mathfrak{R}_{f,\perp}^{\mathbf{h_1}} \neq \emptyset$, as already stated in (3.23).

Since $\xi_{(h_1+1)}$ is nilpotent, the kernel is spanned by the $\widehat{\Theta}_{[h_1];\alpha(k_1)}$, $k_1 = 0,\ldots,s_{1,2}-1$, that obey

$$\Delta_{[h_1];\alpha(k_1)}^f = 0, \qquad (\Delta_{[h_1]}^f - \Delta_{[h_1];\alpha(k_1)}^f)\widehat{\Theta}_{[h_1];\alpha(k_1)} := 0 . \tag{3.91}$$

Restricting our analysis to the critically massless values, i.e.

$$I = 1 : \Delta_{[h_1]}^{f_1^-} = \widehat{N}_{h_1+1}^{h_1+1} - \widehat{N}_{h_1}^{h_1} , \tag{3.92}$$

$$I > 1 : \Delta^{f_I^-}_{[h_1]} = \widehat{N}^{h_1+1}_{h_1+1} + p_I + 1 - h_1 - \widehat{N}^{p_I+1}_{p_I+1} , \qquad (3.93)$$

it follows that

$$I = 1 : \Delta^{f_1^-}_{[h_1];\alpha(k_1)} = s_2 + k_1 - s_1 + 1 = 0 \quad \text{iff } k_1 = s_{1,2} - 1 , \qquad (3.94)$$

$$I > 1 : \Delta^{f_I^-}_{[h_1];\alpha(k_1)} = s_2 + k_1 + p_I + 1 - h_1 - s_I > 0 \quad \text{for } k_1 = 0, \ldots, s_{1,2} - 1 . \ (3.95)$$

Thus it is only the unitarizable critical value f_1^- that yields a potential module, i.e.

$$\mathfrak{R}_{f_I^-}\Big|_{\mathfrak{g}_\lambda} = \widehat{\mathfrak{S}}_{f_I^-} \oplus \mathfrak{R}'_{f_I^-} , \qquad \mathfrak{R}'_{f_I^-}\Big|_{\mathfrak{g}_\lambda} = \begin{cases} \widetilde{\mathfrak{R}}^{h_1}_{f_1^-} \ni \mathfrak{C}^0_{f_1^-} & I = 1 \\ \mathfrak{C}^0_{f_I^-} & I > 1 \end{cases} . \qquad (3.96)$$

The unitarizable h_1-form potential $\widehat{U}^{\mathbf{h_1}}(\widehat{\Theta}_{[h_1]})$ sits in the $\mathfrak{g}$-type

$$\widehat{\Theta}_{[h_1]} := ([s_1 - 1; h_1 + 1], [s_2; h_2], \ldots, [s_B; h_B]) , \qquad (3.97)$$

which we identify as the ASV gauge potential [2].

The embedding of $\widehat{U}^{\mathbf{h_1}}(\widehat{\Theta}_{[h_1]})$ into $\widehat{\mathbf{X}}^{\mathbf{h_1}}$ is given by

$$\widetilde{\mathfrak{R}}^{\mathbf{h_1}}_{f_1^-} := \text{Ker}(\lambda \Delta^{f_1^-}_{[h_1]} + i\xi_{(h_1+1)}) \cap \mathfrak{R}^{\mathbf{h_1}}_{f_1^-, \perp} \ni \widehat{\mathbf{X}}^{\mathbf{h_1}}_{\text{ASV}} = e^{\frac{i\widehat{\xi}_{(h_1+1)}}{\lambda}} \widehat{U}^{\mathbf{h_1}}(\widehat{\Theta}_{[h_1]}) , \quad (3.98)$$

and the resulting generalized curvature constraint takes the form[8]

$$\widehat{R}^{\mathbf{h_1+1}}_{\text{ASV}} := (\widehat{\nabla} - iN\widehat{\xi}_{(h_1+1)})\widehat{U}^{\mathbf{h_1}} - i\,\widehat{e}_{(1)} \cdots \widehat{e}_{(h_1+1)}\widehat{\mathbb{P}}(h_1 + 1, 1)\widehat{\mathbf{X}}^{\mathbf{0}} \approx 0 . \qquad (3.99)$$

Its pullback to a fixed AdS_D leaf with radius L reads

$$R^{\mathbf{h_1+1}}_{\text{ASV}} := i_L^*\widehat{\nabla}U^{\mathbf{h_1}} - i\, i_L^*(\widehat{e}_{(1)} \cdots \widehat{e}_{(h_1+1)})\widehat{\mathbb{P}}(h_1 + 1, 1)\mathbf{X}^{\mathbf{0}} \approx 0 , \qquad (3.100)$$

where $i_L^*\widehat{e}_{(i)} = e^a\widehat{\beta}_{a,(i)}$ and $i_L^*\widehat{\nabla} = d - \frac{i}{2}\Omega^{AB}\widehat{M}_{AB}$ with $\widehat{M}_{AB}$ acting canonically on $\mathfrak{g}$-types and $\Omega^{AB} = (\omega^{ab}, \lambda e^a)$ being the flat $\mathfrak{g}$-connection.

We stress again that, although Weyl's complete reducibility theorem only allows gluing the infinite-dimensional zero-form module to one module in higher form-degree, the latter need not necessarily be the unitary ASV potential. More precisely, taking different combinations of the fields occurring on the right-hand side of eq. (3.20), possibly together with some zero-forms, it should be possible to find non-unitary ASV potential in form-degree higher that h_1 that will appear directly glued to the corresponding Weyl zero-form module in the reduced equation.

---

[8]The exponential in (3.98) "untwists" the "twisted" covariant derivative in the constraint on $\widehat{\mathbf{X}}^{\mathbf{h_1}}$. The zero-form constraint cannot be untwisted, however, since $(\lambda\Delta^{f_1^-}_{[0]} + i\xi_{(1)})\widehat{\mathbf{X}}^{\mathbf{0}} \approx 0$ implies that $\exp(-\frac{i\xi_{(1)}}{\lambda})\widehat{\mathbf{X}}^{\mathbf{0}}$ is logarithmically divergent.

3.6.2 On σ^--cohomology for unitarizable ASV gauge potential

The constraints $R^{\mathbf{1}} \approx 0$ and $R^{\mathbf{h_1+1}}_{\mathrm{ASV}} \approx 0$ have the form $(\nabla + \sigma_0)X \approx 0$ where $X \in \mathfrak{R}'_{\mathrm{ASV}} := \mathfrak{R}'_{f_1^-} = \widetilde{\mathfrak{R}}^{\mathbf{h_1}}_{f_1^-} \uplus \mathfrak{C}^{\mathbf{0}}$, $\nabla = d - \frac{i}{2}\omega^{ab}\widehat{M}_{ab}$ and $\sigma_0 = \sigma^{\mathbf{1}}_{\mathbf{0,0}} + \sigma^{[h_1+1]}_{\mathbf{0,0}} + \sigma^{[h_1+1]}_{0,[h_1]}$ with

$$\sigma^{\mathbf{1}}_{\mathbf{0,0}} = -ie^a \rho^{\mathbf{0}}_{\mathbf{0,0}}(P_a) = -ie^a(\lambda \widehat{\xi}^B \widehat{M}_{Ba} + \widehat{\beta}_{a,(1)}), \tag{3.101}$$

$$\sigma^{[h_1+1]}_{0,[h_1]} = -ie^a \rho^{[h_1]}_{0,[h_1]}(P_a) = -ie^a \widehat{\xi}^B \widehat{M}_{Ba}, \tag{3.102}$$

$$\sigma^{[h_1+1]}_{\mathbf{0,0}} = -\frac{i}{h_1+1}e^a \rho^{[h_1]}_{\mathbf{0,0}}(P_a|e)$$

$$= -\frac{i}{h_1+1}e^{a_1}\cdots e^{a_{h_1+1}}\beta_{a_1,(1)}\cdots\beta_{a_{h_1+1},(h_1+1)}\mathbb{P}(h_1+1,1). \tag{3.103}$$

The corresponding triangular module $\mathfrak{T}'_{\mathrm{ASV}} = \bigoplus_{q\in\mathbb{Z}}\mathfrak{R}'_q$ where $\mathfrak{R}'_0 := \mathfrak{R}'_{\mathrm{ASV}}$. If e^a is non-degenerate, then the maps $\sigma_q = (-1)^{q(1+\sigma_0)}\sigma_0$ decompose into $\sigma_q = \sigma_q^- + \sigma_q^+$ with respect to the ordering $g : \mathfrak{R}'_q \to \mathbb{N}$ defined by

$$g\left(\mathfrak{R}'^{\mathbf{p}_{\alpha_i}+\mathbf{q}}_q(\Theta_{[p_{\alpha_i}];\alpha_i})\right) := g(\alpha) := \alpha + s_{1,2}, \tag{3.104}$$

where the primary type-setting index $\alpha \in s_{2,1} + \mathbb{N}$ is defined by

$$s_{2,1} \leqslant \alpha \leqslant s_2: \qquad \Theta_{[h_1];\alpha_i} \in \widehat{\Theta}_{[h_1]}\Big|_{\mathfrak{m}}, \qquad |\Theta_{[h_1];\alpha_i}| := |\widetilde{\Theta}| + \alpha + s_{1,2}, \tag{3.105}$$

$$0 \leqslant \alpha: \qquad \Theta_{[0];\alpha_i} \in \mathcal{T}(\overline{\Theta})\Big|_{\mathfrak{m}}, \qquad |\Theta_{[0];\alpha_i}| := |\overline{\Theta}| + \alpha, \tag{3.106}$$

where $\widetilde{\Theta}$ is the smallest $\mathfrak{m}$-type in $\widehat{\Theta}_{[h_1]}$, viz.

$$\widetilde{\Theta} = ([s_1-1;h_1],\Xi), \quad \Xi := ([s_2-1;h_2],\ldots,[s_B-1;h_B]), \tag{3.107}$$

and the secondary type-setting index $i = 1,\ldots,n_\alpha$ takes into the account degeneracies (due to that there are many internal $\mathfrak{m}$-types of fixed rank). The resulting $\mathfrak{R}'_q = \bigoplus_{k\in\mathbb{N}}T'_{k,q}$ where

$$T'_{k,q} := g^{-1}(k) \cap \mathfrak{R}'_q = \bigoplus_{\alpha,g(\alpha)=k}\mathfrak{R}'^{\mathbf{p}_\alpha+\mathbf{q}}_q(\Theta_\alpha) = T'^{\mathbf{0}}_{k,q} \oplus T'^{\mathbf{h_1}}_{k,q}, \tag{3.108}$$

$$T'^{\mathbf{p}}_{k,q} = \left(i_{\theta^{a_1}}\cdots i_{\theta^{a_{p+q}}}\Omega^{\mathbf{p+q}}(U) \otimes R'_{[p];\alpha(g)}\right)\Big|_{\mathfrak{m}}, \qquad p = 0, h_1, \tag{3.109}$$

$$R'_{[p];\alpha} = \bigoplus_{i=1}^{n_\alpha}\Theta_{[p];\alpha_i}. \tag{3.110}$$

The space $R'_{[h_1];\alpha(g)}$ ($g \in \{0,\ldots,s_1\}$) is obtained from $R'_{[h_1];\alpha(0)}$ by inserting g cells below the first block while adhering to the rules of Young diagrams. This amounts to

$$R'_{[h_1];\alpha(g)} = ([s_1-1;h_1],\Xi \diamond (g)), \tag{3.111}$$

$$\Xi \diamond (g) := \left\{\Xi' \in \Xi \otimes (g) : |\Xi'| = |\Xi| + g, \text{ width}(\Xi') \leqslant s_1 - 1\right\}, \tag{3.112}$$

$$= \left\{\Xi' \in \Xi \widetilde{\otimes}(g) : \text{ width}(\Xi') \leqslant s_1 - 1\right\}, \tag{3.113}$$

where $\otimes$ is the direct product of $\mathfrak{m}$-tensors and $\widetilde{\otimes}$ is the direct product of $\mathfrak{sl}(D)$-tensors. It follows that

$$g \leqslant s_{1,2} \Rightarrow \diamond = \widetilde{\otimes}. \tag{3.114}$$

such that

$$T'_{[h_1];q,g} \overset{s_{1,2} \geqslant g}{\cong} \bigoplus_{k=0}^{p} \bigoplus_{\substack{k_1 \leqslant h_1, \; k_3 = 0,1 \\ k_2 = k - k_1 - k_3 \geqslant 0 \\ p_1 \leqslant h_1 \\ p_2 = p - k - p_1 \geqslant 0}} \left[\begin{array}{c} \left[\begin{array}{c} [s_1 - 1; h_1 - k_1] \\ [s_1 - 2; k_1] \end{array} \right] \widetilde{\otimes} [p_1] \\ i_{[k_2]} \Xi \; \widetilde{\otimes} (g - k_3) \; \widetilde{\otimes} [p_2] \end{array} \right], \qquad (3.115)$$

where $i_{[k_2]}\Xi$ denotes the direct sum of shapes given by the contraction of k_2 anti-symmetric cells from the shape Ξ.

In what follows we examine the σ^-- cohomology in more detail in the cases $h_1 = 1, 2$.

The example $\Theta = (2, 1)$. The irreducible module carrying the unitary representation $\mathfrak{D}(D - 1; (2, 1))$ is given by

$$\mathfrak{R}'_{\text{ASV}} = \left\{ \underbrace{U^{\mathbf{1}}\widehat{[3]}}_{g=0,1}; \underbrace{C^{\mathbf{0}}[2, 2]}_{g=1}; \underbrace{X^{\mathbf{0}}[3, 2], X^{\mathbf{0}}(3, 2)}_{g=2}; \cdots \right\}. \qquad (3.116)$$

The corresponding triangular module $\mathfrak{T}' = \mathfrak{R}'_{-1} \oplus \mathfrak{R}'_0 \oplus \mathfrak{R}'_1 \oplus \mathfrak{R}'_2 \oplus \cdots$, with variables $\mathfrak{R}'_0 = \mathfrak{R}'$, parameters in $\mathfrak{R}'_{-1} = \left\{ \epsilon^{\mathbf{0}}\widehat{[3]} \right\}$ with $g = 0, 1$, constraints in $\mathfrak{R}'_1 = \left\{ R^{\mathbf{2}}\widehat{[3]}; R^{\mathbf{1}}[2, 2]; R^{\mathbf{1}}[3, 2], R^{\mathbf{1}}(3, 2); \cdots \right\}$ with $g \geqslant 0$, and first level of Bianchi identities in $\mathfrak{R}'_2 = \left\{ Z^{[3]}\widehat{[3]}; \cdots \right\}$ with $g \geqslant 0$.

The non-trivial σ^-- cohomology for $q \leqslant 1$ is a parameter $\epsilon[2]$ at $g = 0$, two fields $\varphi(2, 1)$ and $S(1)$ at $g = 0$, two Proca-like field equations at $g = 0$ and one Labastida-like field equation at $g = 1$. The degree 1 module is "glued" to the degree 0 module via the Weyl tensor $C(2, 1)$ in $T'^{\mathbf{0}}_{q=0,g=1}$ via a constraint in $T'^{\mathbf{1}}_{q=1,g=0}$.

The case $h_1 = 1$, $B \geqslant 2$ and $s_1 - s_2 \geqslant 4$. In this generic case the triangular module (see figure 3)

$$\mathfrak{T}'_{\text{ASV}} = \mathfrak{R}_{-1} \oplus \mathfrak{R}'_0 \oplus \mathfrak{R}'_1 \oplus \mathfrak{R}'_2 \oplus \mathfrak{R}'_3 \oplus \cdots \; \ni \; (\epsilon, X, R, Z, Z_3, \dots), \qquad (3.117)$$

where $\epsilon \in \Omega^{\mathbf{0}}(U)$ for $\alpha < 0$ and $\epsilon \equiv 0$ for $\alpha \geqslant 0$. For $s_1 - s_2 \geqslant 4$ the lowest σ^--chains are

$$g+q=-1 : 0 \hookrightarrow \epsilon^{\mathbf{0}}(R'_{\alpha(0)}) \to 0, \qquad (3.118)$$

$$g+q=0 : 0 \hookrightarrow \epsilon^{\mathbf{0}}(R'_{\alpha(1)}) \to X^{\mathbf{1}}(R'_{\alpha(0)}) \to 0, \qquad (3.119)$$

$$g+q=1 : 0 \hookrightarrow \epsilon^{\mathbf{0}}(R'_{\alpha(2)}) \to X^{\mathbf{1}}(R'_{\alpha(1)}) \to R^{\mathbf{2}}(R'_{\alpha(0)}) \to 0, \qquad (3.120)$$

$$g+q=2 : 0 \hookrightarrow \epsilon^{\mathbf{0}}(R'_{\alpha(3)}) \to X^{\mathbf{1}}(R'_{\alpha(2)}) \to R^{\mathbf{2}}(R'_{\alpha(1)}) \to Z^{\mathbf{3}}(R'_{\alpha(0)}) \to 0, \qquad (3.121)$$

$$g+q=3 : 0 \hookrightarrow \epsilon^{\mathbf{0}}(R'_{\alpha(4)}) \to X^{\mathbf{1}}(R'_{\alpha(3)}) \to R^{\mathbf{2}}(R'_{\alpha(2)}) \to Z^{\mathbf{3}}(R'_{\alpha(1)}) \to Z^{\mathbf{4}}_3(R'_{\alpha(0)}) \to 0, \qquad (3.122)$$

where the $\mathfrak{m}$-content of the parameters is given by $\epsilon^{\mathbf{0}}(R'_{\alpha(g)}) \in \left[\begin{array}{c} s - 1 \\ \Xi \; \widetilde{\otimes}(g) \end{array} \right]$. The chain with $g + q = -1$ contains the differential gauge parameter given by

$$H_{-1,0}(\sigma^-) \ni \epsilon[(s_1 - 1); \Xi]. \qquad (3.123)$$

The chain with $g + q = 0$, where

$$X^{\mathbf{1}}(R'_{\alpha(0)}) \in \begin{bmatrix} s_1 \\ \Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \end{bmatrix} , \tag{3.124}$$

leaves dynamical tensor gauge fields in

$$H_{0,0}(\sigma^-) \ni \varphi[(s_1); \Xi] \oplus A[(s_1 - 2); \Xi] \oplus S[(s_1 - 1); i_{[1]}\Xi] \tag{3.125}$$

where φ, A and S denote the three Lorentz-irreps that occur in the dynamical metric-like field. We shall use similar notation below. The chain with $g + q = 1$, where

$$X^{\mathbf{1}}(R'_{\alpha(1)}) \in \begin{bmatrix} s_1 \\ \Xi \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \widetilde{\otimes}(1) \end{bmatrix}$$
$$\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \end{bmatrix} , \tag{3.126}$$

$$R^{\mathbf{2}}(R'_{\alpha(0)}) \in \begin{bmatrix} s_1 \\ \Xi \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \end{bmatrix}$$
$$\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \end{bmatrix} , \tag{3.127}$$

leaves Proca-like field equations in the Lorentz-irreps

$$H_{1,0}(\sigma^-) \ni P_A[(s_1 - 2); i_{[1]}\Xi] \oplus P_S[(s_1 - 1); i_{[2]}\Xi] \oplus P_\varphi[(s_1); i_{[1]}\Xi] . \tag{3.128}$$

The chain with $g + q = 2$, whose content is listed in appendix D, leaves: i) Labastida-like field equations in

$$H_{1,1}(\sigma^-) \ni F_\varphi[(s_1); \Xi] \oplus F_A[(s_1 - 2); \Xi] ; \tag{3.129}$$

and ii) a Bianchi identity for the Proca-like equations, in

$$H_{2,0}(\sigma^-) \ni B[(s_1); i_{[2]}\Xi] . \tag{3.130}$$

The chain with $q + g = 3$, whose content is listed in appendix D, leaves Noether[9]/Bianchi identities in

$$H_{2,2}(\sigma^-) \ni N_\varphi^1[(s_1 - 1); \Xi] \oplus N_\varphi^2[(s_1); i_{[1]}\Xi] \oplus N_A[s_1 - 2; i_{[1]}\Xi] . \tag{3.131}$$

Thus the dynamical system consists of a parameter ϵ; fields φ, A and S; Proca-like equations of motion of the schematic form $P_\varphi := \nabla_\Xi \phi + \overline{\nabla}^{(1)}S \approx 0$, $P_A := \nabla_\Xi A + \nabla_{(1)}S \approx 0$, $P_S := \nabla_\Xi S \approx 0$; and Labastida-like field equations $F_\varphi \approx 0$ and $F_A \approx 0$ containing the d'Alembertians of ϕ and A, respectively. In the above ∇_Ξ denotes all possible divergencies in Ξ. The parameter can be used to gauge away $\nabla_{(1)}\varphi$, so that $\nabla_{(1)}P_\varphi \approx 0$ implies a mass-shell condition for S. Since all field are now on-shell, the divergencies $\nabla_\Xi \epsilon$ and $\nabla_{(1)}\epsilon$, respectively, of the residual parameter ϵ can be used to remove S and A, leaving a transverse on-shell Lorentz tensor ϕ.

---

[9]Strictly speaking, one should use the terminology *Noether identity* only in case one has an action principle.

	$\mathfrak{R}_{-1}$	$\mathfrak{R}_0$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$
$R_{\alpha(0)}$	$\star$	$\blacklozenge\,\blacklozenge$ $\blacklozenge$	$\bullet\,\bullet$ $\bullet$	$\blacktriangle$	$\blacktriangledown$
$R_{\alpha(1)}$			$\blacklozenge$ $\blacklozenge$	$\bullet\,\bullet$ $\bullet$	
$R_{\alpha(2)}$					
$R_{\alpha(3)}$					
$R_{\alpha(4)}$					
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$R_{\alpha(s_{1,2})}$			$\square$		
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 3. The σ^- cohomology in the case of $h_1 = 1$: i) The $\star$ is the differential gauge parameter; ii) the $\blacklozenge$ at $q = 0$ are the dynamical fields; iii) the $\bullet$ at $q = 1$ are the Proca-like first-order field equations; iv) the $\blacklozenge$ at $q = 1$ are the Einstein-Fronsdal-Labastida-like second-order field equations; v) the $\bullet$ at $q = 2$ are Noether/Bianchi identities; vi) the $\blacktriangle$ and $\blacktriangledown$ are higher Bianchi identities. The $\square$ is the primary Weyl tensor which "glues" the potential module to the Weyl zero-form module. While it is not part of the total $\sigma-$ cohomology, it is part of the σ^--cohomology restricted to the potential module.

The case $h_1 = 2$, $B \geqslant 2$, $s_1 - s_4 \geqslant 4$. Here the triangular module (see figure 3)

$$\mathfrak{T}'_{\mathrm{ASV}} = \mathfrak{R}'_{-2} \oplus \mathfrak{R}'_{-1} \oplus \mathfrak{R}'_0 \oplus \mathfrak{R}'_1 \oplus \mathfrak{R}'_2 \oplus \mathfrak{R}'_3 \oplus \cdots \;\ni\; (\eta, \epsilon, X, R, Z, Z_3, \dots)\,, \quad (3.132)$$

and one can show that if $s_1 - s_2 \geqslant 4$ then the dynamical system contains ($s := s_1$) parameters $\epsilon(s, s-1; \Xi)$, $\epsilon_S(s-1, s-1; i_{[1]}\Xi)$ and $\epsilon_A(s-1, s-2; \Xi)$; fields $\varphi(s, s; \Xi)$, $\varphi_A(s, s-2; \Xi)$, $\varphi_S(s, s-1; i_{[1]}\Xi)$, $S_A(s-1, s-2; i_{[1]}\Xi)$, $S(s-1, s-1; i_{[2]}\Xi)$ and $A(s-2, s-2; \Xi)$; Proca-like equations $P_\varphi(s, s; i_{[1]}\Xi) := \nabla_\Xi \varphi + \overline{\nabla}^{(2)} \varphi_S \approx 0$, $P_{\varphi_A}(s, s-2; i_{[1]}\Xi) := \nabla_\Xi \varphi_A + \nabla_{(2)}\varphi_S + \overline{\nabla}^{(1)} S_A \approx 0$, $P_{\varphi_S}(s, s-1; i_{[2]}\Xi) := \nabla_\Xi \varphi_S + \overline{\nabla}^{(1)} S \approx 0$, $P_{S_A}(s-1, s-2; i_{[2]}\Xi) := \nabla_\Xi S_A + \nabla_{(2)} S \approx 0$, $P_A(s-2, s-2; i_{[1]}\Xi) := \nabla_\Xi A + \nabla_{(1)} S_A \approx 0$ and $P_S(s-1, s-1; i_{[3]}\Xi) := \nabla_\Xi S \approx 0$; and Labastida-like field equations $F_\varphi(s, s; \Xi) \approx 0$, $F_{\varphi_A}(s, s-2; \Xi) \approx 0$ and $F_A(s-2, s-2; \Xi) \approx 0$. The parameters ϵ, ϵ_S and ϵ_A, respectively, can be used to gauge away $\nabla_{(2)}\varphi$, $\nabla_{(1)}\varphi_S$ and $\nabla_{(2)}\varphi_A$, whereafter all fields are on-shell. The on-shell gauge parameters can then be used to gauge away all fields except φ.

$g' = g + 2$	$g' = 0$	$g' = 1$	$g' = 2$	$g' = 3$
$\mathfrak{R}_{-2}$	—	$\eta_{\mathbf{0}}^{\bullet}$	$\eta_{\mathbf{0}}^{[1]}$	
$\mathfrak{R}_{-1}$	$\epsilon_{\mathbf{0}}^{\bullet}$	$\epsilon_{\mathbf{0}}^{[1]}$, $\epsilon_{\mathbf{1}}^{\bullet}$	$\epsilon_{\mathbf{1}}^{[1]}$, $\kappa_{\mathbf{0}}^{[2]}$	$\kappa_{\mathbf{0}}^{[3]}$
$\mathfrak{R}_0$	$Y_{\mathbf{1}}^{\bullet}$	$Y_{\mathbf{1}}^{[1]}$, $X_{\mathbf{2}}^{\bullet}$, $Y_{\mathbf{0}}^{[2]}$	$Y_{\mathbf{0}}^{[3]}$, $X_{\mathbf{2}}^{[1]}$, $X_{\mathbf{1}}^{[2]}$	$X_{\mathbf{1}}^{[3]}$

Table 1. The set of p-form fields obtained upon radial reduction of the p-forms ($p > 0$) associated with the Skvortsov module starting from $\widehat{\varphi}(\Lambda = 0; \widehat{\Theta})$ with $\widehat{\Theta} = ([2;1], [1;1])$. All the fields take value in Lorentz-irreducible shapes. The relation between the two different gradings used in section 3.6.1 and in figure is $g = g' - 2$. The grading g is associated with the ASV potential whereas the g' grading is associated with all the fields obtained upon radial reduction from $D + 1$ to D.

3.7 Stückelberg fields and flat limit

In this section we first look at some examples of the unfolded module $\mathfrak{R}_{f_I^-}$ defined in (3.96) which we, based on the analysis performed so far, claim consists of the ASV module plus the unfolded Stückelberg fields minus the Weyl zero-form of the Stückelberg field χ_I associated with the Ith block — see the discussion in section 3.3.2 — that is projected away by the subsidiary constraint (3.72). We then argue that $\mathfrak{R}_{f_I^-}$ has a smooth flat limit in the sense of the BMV conjecture albeit with additional topological p-forms in flat space coming from the Stückelberg sector in AdS.

3.7.1 The example of $\Theta = (2, 1)$

The σ^--cohomology of the triangular module associated with $\mathfrak{R}_{f_I^-}$ with $I = 1$ is depicted in table 2, where we have assigned a new grading g' (see caption) to all the radially reduced unfolded variables, including those associated with the various Stückelberg fields. All these fields are thus various components of the $\mathfrak{iso}(2, D - 1)$-irreducible Skvortsov module associated with $\widehat{\varphi}(\widehat{\Theta})$.

We note that the cohomology contains two antisymmetric rank-2 objects that could form a trivial pair, namely the cohomologically nontrivial gauge parameter and the zero-form $Y_{\mathbf{0}}^{[2]}$ in $\mathfrak{R}^{\mathbf{0}}$. These quantities do not form a trivial pair because the field equation for $Y_{\mathbf{0}}^{[2]}$ loses its source precisely in the unitary critically massless limit. We interpret the field $Y_{\mathbf{0}}^{[2]}$ as a zero-mode for $\chi_1[2]$ that remains upon imposing the subsidiary condition on the primary Weyl tensor $C_{\chi_1}(2, 1) := \overline{\nabla}^{(1)}\chi_1[2] - \text{traces} \approx 0$.

3.7.2 The example of $\Theta = (3, 1)$

Let us consider the subsector $\mathfrak{R}_f(\Lambda; \Theta) \subset \mathfrak{R}_f$ obtained by constrained radial reduction of the gauge field $\widehat{\varphi}(\Lambda = 0; \widehat{\Theta})$ with $\widehat{\Theta} = ([3; 1], [1; 1])$.

The $p-$form sector thus obtained consists of the fields listed in table 3.

The irreducible module carrying the unitary representation $\mathfrak{D}(D; (3, 1))$ is given by

$$\mathfrak{R}' = \left\{ \underbrace{U^{\mathbf{1}}[\widehat{3, 2}]}_{g = 0,1,2}; \underbrace{C^{\mathbf{0}}[2, 2]}_{g = 2}; \underbrace{X^{\mathbf{0}}[3, 2], X^{\mathbf{0}}(3, 2)}_{g = 3}; \cdots \right\} . \tag{3.133}$$

Table 2. The σ^--cohomology of the unitary $(2,1)$ gauge field in AdS_D. The solid shapes represent the cohomology for the dynamical field $\varphi(2,1)$. The dashed shapes represent the cohomology for the closed Weyl zero-form $Y^{\mathbf{0}}\widehat{[3]}$. For the definition of the grading g', see caption of table 1.

	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$\mathfrak{R}_{-2}$	—	$\zeta_0^{\bullet}$	$\zeta_0^{[1]}$	$\zeta_0^{[1,1]}$	—	—
$\mathfrak{R}_{-1}$	$\epsilon_0^{\bullet}$	$\epsilon_0^{[1]}$, $\epsilon_1^{\bullet}$	$\epsilon_0^{[1,1]}$, $\epsilon_1^{[1]}$, $\eta_0^{[2]}$	$\epsilon_1^{[1,1]}$, $\eta_0^{[2,1]}$, $\kappa_0^{[2,1]}$, $\eta_0^{[3]}$	$\kappa_0^{[2,2]}$, $\eta_0^{[3,1]}$, $\kappa_0^{[3,1]}$	$\kappa_0^{[3,2]}$
$\mathfrak{R}_0$	$Y_1^{\bullet}$	$Y_1^{[1]}$, $X_2^{\bullet}$, $Y_0^{[2]}$	$Y_1^{[1,1]}$, $X_2^{[1]}$, $X_1^{[2]}$, $Y_0^{[3]}$, $Y_0^{[2,1]}$, $\widetilde{Y}_0^{[2,1]}$	$X_1^{[2,1]}$, $Y_0^{[3,1]}$, $X_2^{[1,1]}$, $X_1^{[3]}$, $\widetilde{X}_1^{[2,1]}$, $\widetilde{Y}_0^{[3,1]}$, $\widetilde{Y}_0^{[2,2]}$	$\widetilde{X}_1^{[2,2]}$, $\widetilde{X}_1^{[3,1]}$, $\widetilde{Y}_0^{[3,2]}$, $X_1^{[3,1]}$	$X_1^{[3,2]}$

Table 3. The set of p-form fields obtained upon dimensional reduction of $\widehat{\varphi}(\Lambda=0;\widehat{\Theta})$ with $\widehat{\Theta} = ([3;1],[1;1])$. All the fields take value in Lorentz-irreducible shapes.

Table 4. σ^--cohomologies for the unitary, massless, spin-$(3,1)$ field in AdS_D spacetime.

3.7.3 Smooth flat limit

To repeat, the analysis so far shows that in the unitary case the radial reduction (3.10) followed subsidiary constraint (3.72) lead to the following reducible $\mathfrak{so}(2, D-1)$-module:

$$\mathfrak{R}_{f_1^-}(\Lambda; \Theta) = \mathfrak{S}_{f_1^-}(\Lambda; \Theta) \oplus \mathfrak{R}'_{f_1^-}(\Lambda; \Theta) \,, \tag{3.134}$$

$$\mathfrak{R}'_{f_1^-}(\Lambda; \Theta) = \widetilde{\mathfrak{R}}^{\mathbf{h_1}}_{\mathrm{ASV}} \Subset \mathfrak{C}^{\mathbf{0}}(\Lambda; \overline{M}_1^2; \overline{\Theta}_1) \,, \tag{3.135}$$

where $\mathfrak{S}(\Lambda; \Theta)$ is a massively contractible cycle (see section I.4.4.4) for $\Lambda \neq 0$ containing the BMV Stückelberg fields as well as the frozen (see section I.5.2) Stückeberg fields associated with Ith block (see section 3.3.2).

The above reducible module has the *smooth limit*

$$\mathfrak{R}_{f_1^-}(\Lambda; \Theta) \xrightarrow{\lambda \to 0} \mathfrak{R}_{\mathrm{extra}}(\Lambda=0; \Theta) \ \cup \ \mathfrak{R}_{\mathrm{BMV}}(\Lambda=0; \Theta) \,, \tag{3.136}$$

$$\mathfrak{R}_{\mathrm{BMV}}(\Lambda=0; \Theta) := \bigoplus_{\Theta' \in \Sigma^1_{\mathrm{BMV}}(\Theta)} \mathfrak{R}_{\mathrm{Skv}}(\Lambda=0; \Theta') \,, \tag{3.137}$$

where $\mathfrak{R}_{\mathrm{Skv}}(\Lambda=0; \Theta')$ are the Skvortsov modules predicted by the BMV conjecture and the complement $\mathfrak{R}_{\mathrm{extra}}$ contains a finite set of topological fields. For a fixed Θ and $\Lambda = 0$, one can show that $Y^{\mathbf{h_1}-\mathbf{1}}$ and $\widehat{\xi}_{(h_1+1)} X^{\mathbf{h_1}}$ still form a massively contractible cycle, so that the flat-space potential modules $\widetilde{\mathfrak{R}}(\Lambda = 0; \Theta')$ do not contain any elements of form degree less than h_1, nor of form degree h_1 with first block smaller than that of Θ.

The Weyl zero-form module $\mathfrak{C}^{\mathbf{0}}(\Lambda; \overline{M}_1^2; \overline{\Theta}_1) = \Omega^{\mathbf{0}}(U) \otimes \mathcal{T}(\Lambda; M_1^2; \overline{\Theta}_1)$ has the limit

$$\mathcal{T}(\Lambda; \overline{M}_1^2; \overline{\Theta}_1) \xrightarrow{\lambda \to 0} \bigoplus_{\Theta' \in \Sigma^1_{\mathrm{BMV}}(\Theta)} \mathcal{T}(\Lambda=0; M^2=0; \overline{\Theta}') \,, \tag{3.138}$$

which together with harmonic expansion shows that the unitary massless lowest-weight space representation of $\mathfrak{so}(2, D-1)$ contracts to the direct sum of massless irreps of $\mathfrak{iso}(1, D-1)$ in accordance with the BMV conjecture.

Finally, we note that it should be possible to project away the aforementioned frozen field content for $\Lambda \neq 0$ without affecting the smoothness of the flat limit, which we leave for future studies.

4 Conclusion

In the present paper we studied the BMV conjecture [19] at the level of the field equations by extending the unfolding analysis carried out by Skvortsov in [3] to the AdS_D background. To this end, we reformulated the equations of [3] by using an oscillator formalism. Certain operators were constructed, the so-called cell operators, which were found to be very useful for an alternative proof of the consistency of Skvortsov's equations.

We then proceeded with the following steps: We started from the reformulation of Skvortsov's unfolded equations for a mixed-symmetry gauge field $\widehat{\varphi}(\widehat{\Theta})$ in $(D + 1)$-dimensional flat space with signature $(2, D-1)$ and radially reduced the $(D+1)$-dimensional

unfolded fields to AdS_D. Then we constrained the Lie derivatives of the fields along a radial vector field [see eqs. (3.10), (3.12) and (3.17)]. Next, we constrained the $(p_I + 1)$th row of the internal indices carried by the zero-forms [see eq. (3.72)] and verified that the generalized Weyl tensor of $\varphi(\Theta)$ carries Metsaev's unitary representation $\mathfrak{D}(e_0; \Theta)$.

In particular, we were able to prove the BMV conjecture in the case of mixed-symmetry gauge fields whose corresponding Young diagrams possess at most four rows. The nontrivial consistency of the constraints imposed on the generalized Weyl tensors is a good sign that these constraints are correct for arbitrary mixed-symmetry gauge fields. In a future work we would like to further study the consistency of our constraints in the general case. For this, it is crucial to have a better understanding of the cell operators and their commutation relations. Also of interest is to study further the p-form sector ($p > 0$) of the unfolded system in AdS_D that we displayed in the present work, in particular the precise expression of the constraints that would enable one to project out the frozen Stückelberg fields.

In relation with the previous issue, it would be very interesting to make a precise link between these p-forms and the gauge fields needed for a first-order action formulation of arbitrary mixed-symmetry fields in AdS_D along the lines proposed by Zinoviev, see [7] for the cases where the shape associated with the field is a long hook with one cell in the second row.

As shown in appendix E, *generic* mixed-symmetry fields cannot be seen as singleton composites, though certain long-hook fields arise in tensor products of two spin-1/2 fermionic singletons [46]. The oscillator realization of the constraints in our radial-reduction construction does not appear to allow for a strict factorization in terms of subsets of unconstrained oscillators.

This non-factorization property maybe is an artefact of our construction and it would be very important, we believe, to investigate about an abstract enveloping-algebra approach to the lowest-weight modules corresponding to generic mixed-symmetry fields.

Acknowledgments

A   Notation and conventions

The direct sum of two vector spaces is written as $\mathfrak{A} \uplus \mathfrak{B}$. If $\mathfrak{l}$ is a Lie algebra (or more generally an associative algebra) then the decomposition of an $\mathfrak{l}$-module $\mathfrak{R}$ under a subalgebra $\mathfrak{k} \subseteq \mathfrak{l}$ is denoted by $\mathfrak{R}|_{\mathfrak{k}}$. A module $\mathfrak{R}$ containing an invariant subspace $\mathfrak{I}$, an ideal, is said

to be either (i) indecomposable if the complement of $\mathfrak{I}$ is not invariant in which case one writes $\mathfrak{R}|_{\mathfrak{l}} = \mathfrak{I} \Subset (\mathfrak{R}/\mathfrak{I})$; or (ii) decomposable if both $\mathfrak{I}$ and $\mathfrak{R}/\mathfrak{I}$ are invariant in which case one writes $\mathfrak{R}|_{\mathfrak{l}} = \mathfrak{I} \oplus (\mathfrak{R}/\mathfrak{I})$.

Infinite-dimensional modules can be presented in many ways depending on how they are sliced under various subalgebras. If $\mathfrak{k} \subset \mathfrak{l}$ one refers to finite-dimensional $\mathfrak{k}$-irreps with non-degenerate bilinear forms as $\mathfrak{k}$-types, which we denote by Θ_α, Θ_{α_i} etc. labeled by indices α, α_i etc.. Correspondingly, if there exists a slicing $\mathfrak{R}|_{\mathfrak{k}}$ consisting of $\mathfrak{k}$-types then we refer to such expansions as an $\mathfrak{k}$-typesetting of $\mathfrak{R}$. In particular, we refer to finite-dimensional Lorentz-irreps as Lorentz types (that will be tensorial in this paper). In unfolded dynamics one may view typesetting as local coordinatizations of infinite-dimensional target spaces for unfolded sigma models. We set aside issues of topology.

Young diagrams, or row/column-ordered shapes, with m_i cells in the ith row/column, $i = 1, \ldots, n$ are labeled by $(m_0, \ldots, m_{n+1})$ and $[m_0, \ldots, m_{n+1}]$ where $m_i \geqslant m_{i+1}$ and $m_0 := \infty$ and $m_{n+1} := 0$. We let $\mathbb{P}_\Theta$ denote Young projections on shape Θ. We also use the block-notation

$$([s_1; h_1], [s_2; h_2], \ldots, [s_B; h_B]) := (\underbrace{m_1, \ldots, m_{h_1}}_{=s_1}, \underbrace{m_{h_1+1}, \ldots, m_{h_1+h_2}}_{=s_2} \ldots), \qquad (A.1)$$

for a shape with B rectangular blocks of lengths $s_I > s_{I+1}$ and heights $h_I \geqslant 1$, $I = 1, 2, \ldots, B$. The space of shapes $\mathcal{S}$ forms a module, the Schur module, for the universal Howe-dual algebra $\mathfrak{sl}(\infty)$, obtained as a formal limit of $\mathfrak{sl}(\nu_\pm)$ acting in the spaces $\mathcal{S}^\pm_{\nu_\pm}$ of shapes with total height $p_B := \sum_{I=1}^B h_I \leqslant \nu_+$ ($\mathfrak{sl}(D)$-types in symmetric bases) or widths $s_1 \leqslant \nu_-$ (($\mathfrak{sl}(D)$-types in anti-symmetric bases). Extension to traceless Lorentz tensors leads to Howe-dual algebras $\mathfrak{sp}(2\nu_+)$ and $\mathfrak{so}(\nu_-)$, with formal limits $\mathfrak{sp}(2\infty)$ and $\mathfrak{so}(2\infty)$, respectively.

The Schur module $\mathcal{S}$ can be treated *explicitly* by using "cell operators" $\beta_{a,(i)}$ and $\bar{\beta}^{a,(i)}$ defined (see Paper II) to act faithfully in $\mathcal{S}$ by removing or adding, respectively, a cell containing the $\mathfrak{sl}(D)$-index a in the ith row. Schematically,

$$\bar{\beta}^{a,(i)}(m_1, \ldots, m_i, \ldots, m_n) = (m_1, \ldots, m_i + 1, \ldots, m_n),$$
$$\beta_{a,(i)}(m_1, \ldots, m_i, \ldots, m_n) = (m_1, \ldots, m_i - 1, \ldots, m_n).$$

Similarly, $\beta_{a,[i]}$ and $\bar{\beta}^{a,[i]}$, respectively, remove and add an a-labeled box in the ith column.

We let $\widehat{\mathfrak{g}}$ denote the real form of $\mathfrak{so}(D+1)$ with metric $\eta_{AB} = \operatorname{diag}(\sigma, \eta_{ab})$ where $\sigma = \pm 1$ and $\eta_{ab} = (-1, \delta_{rs})$, and with generators $\widehat{M}_{AB}$ obeying the commutation rules

$$[\widehat{M}_{AB}, \widehat{M}_{CD}] = 2i\, \eta_{C[B}\widehat{M}_{A]D} - 2i\, \eta_{D[B}\widehat{M}_{A]C}. \qquad (A.2)$$

We let $\mathfrak{m} := \mathfrak{so}(1, D-1)$ and $\mathfrak{s} := \mathfrak{so}(D-1)$ denote the "canonical" Lorentz and spin subalgebras, respectively, with generators M_{ab} and M_{rs}. We let $\mathfrak{g}_\lambda := \mathfrak{m} \Subset \mathfrak{p}$ where $\mathfrak{p}$ is spanned by the transvections[10] obeying

$$[P_a, P_b] = i\lambda^2 M_{ab}, \quad [M_{ab}, P_c] = 2i\eta_{c[b}P_{a]}. \qquad (A.3)$$

---

[10]We are here abusing a standard terminology used in the context of symplectic algebras, the only point being to make clear the distinction between the cases where the generators $\{P_a\}$ are commuting or not.

If $\lambda^2 = 0$ then $\mathfrak{g}_\lambda \cong \mathfrak{iso}(1, D-1)$ and if $\lambda^2 \neq 0$ then $\mathfrak{g}_\lambda \cong \widehat{\mathfrak{g}}$ with $\sigma = -\lambda^2/|\lambda^2|$, the isometry algebras of AdS_D ($\sigma = -1$) and dS_D ($\sigma = 1$) with radius $L_{\mathrm{AdS}} := L$ and $L_{\mathrm{dS}} := -i\,L$, respectively, where $L := \lambda^{-1}$ is assumed to be real for AdS_D and purely imaginary for dS_D. The $\mathfrak{g}_\lambda$-valued connection Ω and curvature $\mathcal{R}$ are defined as follows

$$\Omega := e + \omega := -i\left(e^a\,P_a + \frac{1}{2}\,\omega^{ab}\,M_{ab}\right), \tag{A.4}$$

$$\mathcal{R} := d\Omega + \Omega^2 \;=\; -i\left[T^a P_a + \frac{1}{2}(R^{ab} + \lambda^2 e^a e^b)\,M_{ab}\right], \tag{A.5}$$

$$T^a := de^a + \omega^a{}_b\,e^b, \quad R^{ab} := d\omega^{ab} + \omega^a{}_c\,\omega^c{}_b, \tag{A.6}$$

and are associated with a cosmological constant $\Lambda = -\frac{(D-1)(D-2)}{2}\lambda^2$. The Lie derivative along a vector field ξ is $\mathcal{L}_\xi := d\,i_\xi + i_\xi\,d$ and we use conventions where the exterior total derivative d and the inner derivative i_ξ act from the left. If the frame field e^a is invertible we define the inverse frame field θ^a by $i_{\theta^a}e^b = \eta^{ab}$.

We use weak equalities $\approx$ to denote equations that hold on the constraints surface. In the maximally symmetric backgrounds $\mathcal{R} \approx 0$ the connection Ω can be frozen to a fixed background value, breaking the diffeomorphisms down to isometries $\delta_{\epsilon(\xi)}$ with Killing parameters $\epsilon(\xi) = i_\xi(e + \omega)$ obeying $\delta_{\epsilon(\xi)}(e + \omega) \approx \mathcal{L}_\xi(e + \omega) = 0$ (one has $\mathcal{L}_\xi e^a = \delta_{\epsilon(\xi)}e^a + i_\xi T^a$ where $\delta_{\epsilon(\xi)}e^a = \nabla\epsilon^a - \epsilon^{ab}e_b$ with $\epsilon^a = i_\xi e^a$, $\epsilon^{ab} = i_\xi\omega^{ab}$ and $\nabla := d - \frac{i}{2}\omega^{ab}M_{ab}$).

We use $\mathfrak{D}^\pm(\pm e_0; \Theta_0)$ to denote lowest-weight $(+)$ and highest-weight $(-)$ modules of $\mathfrak{g}_\lambda$ that are sliced under its maximal compact subalgebra $\mathfrak{h} \cong \mathfrak{so}(2)\oplus\mathfrak{so}(D-1)$ into $\mathfrak{h}$-types $|e; \theta\rangle^\pm$. In compact basis, the $\mathfrak{so}(2, D-1)$ algebra reads

$$M_{0r} = \frac{1}{2}(L_r^+ + L_r^-), \qquad P_r = \frac{i\lambda}{2}(L_r^+ - L_r^-), \qquad E = \lambda^{-1}P_0, \tag{A.7}$$

$$[L_r^-, L_s^+] = 2iM_{rs} + 2\delta_{rs}E, \quad [E, L_r^\pm] = \pm L_r^\pm, \qquad [M_{rs}, L_t^\pm] = 2i\delta_{t[s}L_{r]}^\pm. \tag{A.8}$$

By their definition, the modules $\mathfrak{D}^\pm(\pm e_0; \Theta_0)$ are the irreps obtained by factoring out all proper ideals in the generalized Verma module generated from a unique lowest-energy $(+)$ or highest-energy $(-)$ state $|\pm e_0; \Theta_0\rangle^\pm$ with E-eigenvalue $\pm e_0$. We let $\mathfrak{D}(e_0; \Theta_0) := \mathfrak{D}^+(e_0; \Theta_0)$ and $|e; \theta\rangle := |e; \theta\rangle^+$. The generalized Verma module is irreducible for generic values of e_0, i.e. singular vectors arise only for certain critical values related to Θ_0.

In unfolded field theory the mass-square M^2 of an unfolded Lorentz tensor field $\phi(\Theta)$ (dynamical field, Weyl tensor, ...) carrying a $\mathfrak{g}_\lambda$-irrep ($\Lambda \neq 0$) with representation ρ, is the eigenvalue of

$$-\rho(P^a P_a) \;\equiv\; \lambda^2\rho\left(\frac{1}{2}M_{AB}M^{AB} - \frac{1}{2}M_{ab}M^{ab}\right). \tag{A.9}$$

In the case of $\Lambda < 0$ one sometimes deals with harmonic expansions involving lowest-weight spaces where

$$C_2[\mathfrak{g}_\lambda|\mathfrak{D}(e_0; \Theta_0)] = e_0[e_0 - 2(\epsilon_0 + 1)] + C_2[\mathfrak{s}|\Theta_0], \quad \mathfrak{s} := \mathfrak{so}(D-1), \quad \epsilon_0 := \frac{1}{2}(D-3) \tag{A.10}$$

leading to the mass formula

$$L^2 M^2 = e_0[e_0 - 2(1 + \epsilon_0)] + C_2[\mathfrak{s}|\Theta_0] - C_2[\mathfrak{m}|\Theta]. \tag{A.11}$$

We let $\mathcal{T}^{\pm}_{(i)}(\Theta^{\pm})$ denote $\mathfrak{iso}(1,D-1)$-irreps with (a) largest and smallest $\mathfrak{m}$-types Θ^+ and Θ^-, respectively; and (b) translations represented by $\rho^+_{(i)}(P_a) = \beta_{a,(i)}$ and $\rho^-_{(i)}(P_a) = \bar{\gamma}^{a,(i)}$ (the trace-corrected cell creation operator) for fixed $i \geqslant 1$. As a special case $\mathcal{T}^-_{(1)}(\Theta^-) \cong \mathcal{T}^*(\Lambda{=}0; \overline{M}2{=}0; \Theta^-)$, the dual of the twisted-adjoint representation containing a strictly massless primary Weyl tensor.[11] We also let $\mathcal{T}^{\pm}_{(0)}(\Theta) := \Theta$, the irrep consisting of a single $\mathfrak{m}$-type Θ annihilated by P_a.

The translations are nilpotent in $\mathcal{T}^{\pm}_{(i)}(\Theta^{\pm})$ for $i \geqslant 2$ and in $\mathcal{T}^+_{(1)}(\Theta^+)$. Factoring out ideals yields "cut" finite-dimensional modules $\mathcal{T}^{\pm}_{(i),N}(\Theta^{\pm})$ of "depth" $N \geqslant 0$ such that $\left(\rho^{\pm}_{(i),N}(P_a)\right)^n \not\equiv 0$ iff $n \leqslant N$. For $i \geqslant 2$ the duals $\left(\mathcal{T}^{\pm}_{(i)}(\Theta^{\pm})\right)^* \cong \mathcal{T}^{\mp}_{(i),N}(\Theta'^{\mp})$ for some N and $\Theta'^{\mp}$ determined from the shape of $\Theta^{\pm}$. In particular, $(\mathcal{T}^{\pm}_{(i)}(\Theta^{\pm}))^* \cong \mathcal{T}^{\mp}_{(i)}(\Theta^{\mp})$ iff the ith row does not form a block of its own in Θ^+ nor Θ^-.

The $\mathfrak{iso}(1,D-1)$-irreps $\mathcal{T}^{\pm}_{(i)}(\Theta^{\pm})$ with $i \geqslant 2$ and $\mathcal{T}^+_{(1)}(\Theta^+)$ are contractions of $\mathfrak{so}(2,D-1)$-types as follows: the $\mathfrak{so}(2,D-1)$-type $\widehat{\Theta}$ with its canonical representation $\widehat{M}_{AB}$ is isomorphic to twisted representations $\widehat{\Theta}^{\pm}_{(i),\kappa;\lambda}$ with canonical $\rho^{\pm}_{(i),\kappa;\lambda}(M_{ab}) := \widehat{M}_{ab}$ and non-canonical $\rho^+_{(i),\kappa;\lambda}(P_a) := \lambda\,\widehat{\xi}^B \widehat{M}_{Ba} + \kappa\,\beta_{a,(i)}$ and $\rho^-_{(i),\kappa;\lambda}(P_a) := \lambda\,\widehat{\xi}^B \widehat{M}_{Ba} + \kappa\,\bar{\gamma}_{a,(i)}$ where $\widehat{\xi}^2 = -1$ (these are representations for $[P_a, P_b] = i\lambda^2 M_{ab}$ for all values of κ, λ and i). The limit $\lambda \to 0$ at fixed κ yields a reducible $\mathfrak{iso}(1,D-1)$ representation that decomposes into $\mathcal{T}^{\pm}_{(i)}$-plets if $\kappa \neq 0$ and $\mathcal{T}^+_{(0)}$-plets if $\kappa = 0$.

B   Oscillator realizations of classical Lie algebras

B.1   Howe-dual Lie algebras

We denote the classical algebras by

$$\mathfrak{l} := (\mathfrak{gl}(D;\mathbb{C}), \mathfrak{so}(D;\mathbb{C}), \mathfrak{sp}(D;\mathbb{C})), \qquad \epsilon(\mathfrak{l}) = (0, +1, -1), \tag{B.1}$$

where D is assumed to be even for $\epsilon = -1$. Their finite-dimensional representations can be realized using bosonic $(+)$ and fermionic $(-)$ oscillators, corresponding to tensors in manifestly symmetric or anti-symmetric bases for the Young projector, respectively. Omitting the tensor-spinorial representations of $\mathfrak{so}(D;\mathbb{C})$, the oscillators obey

$$[\alpha_{i,a}, \bar{\alpha}^{j,b}] := \alpha_{i,a}\bar{\alpha}^{j,b} + (-1)^{\frac{1}{2}(1\pm 1)}\bar{\alpha}^{j,b}\alpha_{i,a} = \delta^j_i \delta^b_a, \tag{B.2}$$

where $a,b = 1, \ldots, D$ transform in the fundamental representation of $\mathfrak{l}$, and $i = 1, 2, \ldots, \nu_{\pm}$ are auxiliary flavor indices. The oscillator algebras are invariant under the canonical transformations generated by arbitrary Grassmann even polynomials $\varepsilon(\alpha, \bar{\alpha})$, viz.

$$\delta_\varepsilon \alpha_{i,a} = [\varepsilon(\alpha, \bar{\alpha}), \alpha_{i,a}], \qquad \delta_\varepsilon \bar{\alpha}^{i,a} = [\varepsilon(\alpha, \bar{\alpha}), \bar{\alpha}^{i,a}], \tag{B.3}$$

forming an infinite-dimensional Lie algebra with commutator $[\delta_\varepsilon, \delta_{\varepsilon'}] = \delta_{[\varepsilon,\varepsilon']_\star}$. The linear homogeneous canonical transformations form the finite-dimensional subalgebras

$$\mathfrak{l}^+ := \mathfrak{sp}(2D\nu_+; \mathbb{C}), \qquad \mathfrak{l}^- := \mathfrak{so}(2D\nu_-; \mathbb{C}). \tag{B.4}$$

---

[11]In a similar context, see also the recent work [47] where the unfolding of mixed-symmetry fields in flat space was reformulated using BRST-cohomological methods.

These contain $\mathfrak{l}$ together with its Howe dual[12] $\widetilde{\mathfrak{l}}^{\pm}$ which is defined to be the maximal subalgebra of $\mathfrak{l}^{\pm}$ that commutes with $\mathfrak{l}$. One has

$$\mathfrak{l} = \mathfrak{gl}(D;\mathbb{C}): \qquad \widetilde{\mathfrak{l}}^{\pm} = \mathfrak{gl}(\nu_{\pm})\,, \tag{B.5}$$

$$\mathfrak{l} = \mathfrak{so}(D;\mathbb{C}): \qquad \widetilde{\mathfrak{l}}^{+} = \mathfrak{sp}(2\nu_{+};\mathbb{C}), \qquad \widetilde{\mathfrak{l}}^{-} = \mathfrak{so}(2\nu_{-};\mathbb{C})\,, \tag{B.6}$$

$$\mathfrak{l} = \mathfrak{sp}(D;\mathbb{C}): \qquad \widetilde{\mathfrak{l}}^{+} = \mathfrak{so}(2\nu_{+};\mathbb{C})\,, \qquad \widetilde{\mathfrak{l}}^{-} = \mathfrak{sp}(2\nu_{-};\mathbb{C})\,. \tag{B.7}$$

The oscillator realization of the generators of $\mathfrak{l}$ reads

$$\mathfrak{gl}(D;\mathbb{C}): \ M_b^a = \bar{\alpha}^{i,a}\alpha_{i,b}\,, \tag{B.8}$$

$$\mathfrak{so}(D;\mathbb{C}) \text{ and } \mathfrak{sp}(D;\mathbb{C}): \ M_{ab} = 2i\bar{\alpha}^{i,c}J_{c\langle a|}\alpha_{i,|b\rangle}\,, \tag{B.9}$$

with the commutation rules

$$[M_b^a, M_d^c] \ = \ \delta_b^c M_d^a - \delta_d^a M_b^c\,, \qquad [M_{ab}, M_{cd}] \ = \ 4iJ_{\langle c|\langle b}M_{a\rangle|d\rangle}\,, \tag{B.10}$$

where

$$M_{ab} \ = \ M_{\langle ab\rangle} \ := \ -\epsilon M_{ba}\,, \qquad J_{ab} = \epsilon J_{ba}\,, \qquad J^{ab}J_{ac} \ = \ \delta_c^b\,, \tag{B.11}$$

and indices are raised and lowered according to the convention $X^a = J^{ab}X_b$ and $X_a = X^b J_{ba}$. For definiteness, we take $J_{ab} = \eta_{ab}$ of some signature (p,q), $p+q=D$ in the case of $\mathfrak{so}(D;\mathbb{C})$ ($\epsilon = +1$), and $J_{ab} = \Omega_{ab} = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}$ in the case of $\mathfrak{sp}(D;\mathbb{C})$ ($\epsilon = -1$). The oscillator realization of the generators of $\widetilde{\mathfrak{l}}^{\pm}$ reads

$$N_j^i := \frac{1}{2}\{\bar{\alpha}^{i,a}, \alpha_{j,a}\} \equiv \frac{1}{2}(\bar{\alpha}^{i,a}\alpha_{j,a} + \alpha_{j,a}\bar{\alpha}^{i,a})\,, \quad T_{ij} := \alpha_{i,a}\alpha_{j,b}J^{ab}\,, \quad \overline{T}^{ij} := \bar{\alpha}^{i,a}\bar{\alpha}^{j,b}J_{ab}\,. \tag{B.12}$$

Their commutation rules take the form

$$[T_{ij}, \overline{T}^{kl}] = 4\,N_{\langle i}^{\langle k}\delta_{j\rangle}^{l\rangle}\,, \qquad\qquad [N_j^i, N_l^k] = \delta_j^k N_l^i - \delta_l^i N_j^k\,, \tag{B.13}$$

$$[N_j^i, T_{kl}] = -2\,T_{j\langle l}\delta_{k\rangle}^i\,, \qquad\qquad [N_j^i, \overline{T}^{kl}] = 2\,\overline{T}^{i\langle l}\delta_j^{k\rangle}\,, \tag{B.14}$$

where

$$T_{ij} = T_{\langle ij\rangle} \ := \ \pm\epsilon\,T_{ji}\,. \tag{B.15}$$

In the cases of $\epsilon(\mathfrak{l}) = \pm 1$, the above bases exhibit explicitly the three-grading

$$\widetilde{\mathfrak{l}}^{\pm} = T^{-1} \oplus N^0 \oplus \overline{T}^{+1}\,. \tag{B.16}$$

---

[12]A Howe dual pair of Lie algebras is a pair of Lie subalgebras in a Lie algebra which are their mutual centralizers.

B.2 Generalized Schur modules

The oscillator algebra can be realized in various oscillator-algebra modules $\mathcal{M}^\pm$. For given $\mathcal{M}^\pm$, the corresponding *generalized Schur module*

$$\mathcal{S}^\pm := \bigoplus_{\widetilde{\lambda}^\pm} \mathbb{C} \otimes |\widetilde{\lambda}^\pm\rangle\,, \tag{B.17}$$

where $|\widetilde{\lambda}^\pm\rangle$, which we shall refer to as the *Schur states*, are the ground states of $\widetilde{\mathfrak{l}}^\pm$ in $\mathcal{M}^\pm$ with Howe-dual highest weights $\widetilde{\lambda}^\pm = \{\widetilde{\lambda}_i^\pm\}_{i=1}^{\nu_\pm}$. By making a canonical choice of the Borel subalgebra for $\widetilde{\mathfrak{l}}^\pm$, the Schur states can be chosen to obey

$$\forall \epsilon: \qquad (N_j^i - \delta_j^i \widetilde{\lambda}_i^\pm)|\widetilde{\lambda}^\pm\rangle = 0 \qquad \text{for } i \leqslant j \quad (\text{no sum on } i)\,, \tag{B.18}$$

$$\epsilon = \pm 1: \qquad\qquad T_{ij}|\widetilde{\lambda}^\pm\rangle = 0\,. \tag{B.19}$$

We also define the shifted Howe-dual highest weights (see also (B.33) and (B.34) below)

$$\widetilde{w}_i^\pm := \widetilde{\lambda}_i^\pm \mp \frac{D}{2}\,. \tag{B.20}$$

The Schur states $|\widetilde{\lambda}^\pm\rangle$ generate lowest-weight spaces $\mathcal{D}^\pm(\widetilde{\lambda}^\pm)$ of $\widetilde{\mathfrak{l}}^\pm$, and

$$\mathcal{M}^\pm\big|_{\widetilde{\mathfrak{l}}^\pm} = \bigoplus_{\widetilde{\lambda}^\pm \in \widetilde{\Lambda}^\pm} \text{mult}(\widetilde{\lambda}^\pm)\mathcal{D}^\pm(\widetilde{\lambda}^\pm)\,, \tag{B.21}$$

where $\text{mult}(\widetilde{\lambda}^\pm) \in \mathbb{N}$ are multiplicities. For simplicity, we assume that $\mathcal{M}^\pm$ has a non-degenerate inner product and that the $\widetilde{\mathfrak{l}}^\pm$ action on $|\widetilde{\lambda}^\pm\rangle$ does not yield any singular vectors.

By construction an invariant polynomial $C[\mathfrak{l}] \in \mathcal{U}[\mathfrak{l}]$, the enveloping algebra of $\mathfrak{l}$, can be rewritten as an invariant polynomial $C[\widetilde{\mathfrak{l}}^\pm] \in \mathcal{U}[\widetilde{\mathfrak{l}}^\pm]$, and hence assumes a fixed value, $C[\mathfrak{l}|\widetilde{\lambda}^\pm]$ say, in $\mathcal{D}^\pm(\widetilde{\lambda}^\pm)$. Hence, $\mathcal{D}^\pm(\widetilde{\lambda}^\pm)$ decomposes under $\mathfrak{l}$ into

$$\mathcal{D}^\pm(\widetilde{\lambda}^\pm)\big|_{\mathfrak{l}} = \bigoplus_{\lambda \in \Lambda(\widetilde{\lambda}^\pm)} \text{mult}^\pm(\lambda|\widetilde{\lambda}^\pm)\mathcal{D}^\pm(\lambda|\widetilde{\lambda}^\pm)\,, \tag{B.22}$$

where $\Lambda(\widetilde{\lambda}^\pm)$ contains the labels λ of all $\mathfrak{l}$-irreps $\mathcal{D}^\pm(\lambda|\widetilde{\lambda}^\pm)$ obeying $C[\mathfrak{l}|\lambda] = C[\mathfrak{l}|\widetilde{\lambda}^\pm]$ for all invariants C, and $\text{mult}^\pm(\lambda|\widetilde{\lambda}^\pm) \in \{0, 1, \dots\}$ are multiplicities. Consequently,

$$\mathcal{S}^\pm\big|_{\mathfrak{l}} = \bigoplus_{\widetilde{\lambda}^\pm} \bigoplus_{\lambda \in \Lambda(\widetilde{\lambda}^\pm)} \bigoplus_{\mu=1}^{\text{mult}^\pm(\lambda|\widetilde{\lambda}^\pm)} \mathbb{C} \otimes |\lambda|\widetilde{\lambda}^\pm; \mu\rangle\,, \tag{B.23}$$

where $|\lambda|\widetilde{\lambda}^\pm; \mu\rangle \in \mathcal{D}^\pm(\lambda|\widetilde{\lambda}^\pm)$ are the Schur states and the index μ labels the degeneracy of the construction. If $\mathcal{D}(\widetilde{\lambda}^\pm)$ decomposes into finite-dimensional irreps of $\mathfrak{l}$, then the spectrum of invariants $\{C\}$ is sufficiently large to fix λ uniquely in terms of $\widetilde{\lambda}^\pm$, and hence only the multiplicity remains a free parameter. In what follows, one useful Howe-duality relation is that of the quadratic Casimir operators

$$C_2[\mathfrak{gl}(D;\mathbb{C})] := M_b^a M_a^b\,, \quad C_2[\mathfrak{so}(D;\mathbb{C})] := \frac{1}{2}M^{ab}M_{ab}\,, \quad C_2[\mathfrak{sp}(D;\mathbb{C})] := \frac{1}{2}M^{ab}M_{ab} \tag{B.24}$$

that assume the values

$$C_2[\mathfrak{l}|\widetilde{\lambda}^\pm] = \sum_{i=1}^{\nu_\pm} \widetilde{w}_i^\pm(D - \epsilon \pm (\widetilde{w}_i^\pm + 1 - 2i))\,. \tag{B.25}$$

B.3 Fock-space realizations

Acting with the oscillators on a state $|0\rangle$ obeying $\alpha_{i,a}|0\rangle = 0$ yields the standard Fock space

$$\mathcal{F}^{\pm}_{D;\nu_{\pm}} = \bigoplus_{R=0}^{\infty} \mathcal{F}^{\pm}_{D;\nu_{\pm};R}\,, \tag{B.26}$$

where $\mathcal{F}^{\pm}_{D;\nu_{\pm};R} = \left\{ |X\rangle \ : \ \left(\sum_i N_i^i - R \right) |X\rangle \ = \ 0 \right\}$ are subspaces of states of fixed rank R. These spaces have dimensions

$$\dim \mathcal{F}^{\pm}_{D;\nu_{\pm};R} = \frac{1}{R!} D^{\pm}(D^{\pm} \pm 1) \cdots (D^{\pm} \pm (R-1))\,, \qquad D^{\pm} := D\nu_{\pm}\,. \tag{B.27}$$

We note that for fermionic oscillators, $\dim \mathcal{F}^{-}_{D;\nu_{-};R} = 0$ if $R > D^-$ and $\dim \mathcal{F}^{-}_{D;\nu_{-}} = 2^{D^-}$. The Fock space $\mathcal{F}^{\pm}_{D;\nu_{\pm}}$ decomposes under $\mathfrak{l} \times \widetilde{\mathfrak{l}}^{\pm}$ as follows:

$$\mathcal{F}^{\pm}_{D;\nu_{\pm}} \Big|_{\mathfrak{l} \times \widetilde{\mathfrak{l}}^{\pm}} = \bigoplus_{\Delta} \mathrm{mult}^{\pm}_{\epsilon}(\Delta | D; \nu_{\pm}) \, \mathcal{D}^{\pm}(\lambda(\Delta) | \widetilde{\lambda}^{\pm}(\Delta))\,, \tag{B.28}$$

where the sum runs over all possible Young diagrams Δ (including the trivial diagram) and[13]

$$\lambda_i(\Delta) = w_i\,, \qquad\qquad i = 1, \ldots, D\,, \tag{B.32}$$

$$\widetilde{\lambda}_i^+(\Delta) = \frac{D}{2} + w_i\,, \qquad w_i = \widetilde{w}_i^+\,, \qquad i = 1, \ldots, \nu_+\,, \tag{B.33}$$

$$\widetilde{\lambda}_i^-(\Delta) = -\frac{D}{2} + h_i\,, \qquad h_i = \widetilde{w}_i^-\,, \qquad i = 1, \ldots, \nu_-\,, \tag{B.34}$$

with $w_i = w_i(\Delta)$ and $h_i = h_i(\Delta)$ being the number of cells in the ith row and column of Δ, respectively. We shall say that Δ contains a block of height h between the ith and

---

[13]The standard Fock space can be equipped with the positive definite inner product. From $||\widehat{N}_j^i|\Delta\rangle||^2 \geqslant 0$ it follows that $\widetilde{\lambda}_i^{\pm} - \widetilde{\lambda}_j^{\pm} \geqslant 0$ if $i < j$ with equality iff $N_j^i|\Delta\rangle = N_i^j|\Delta\rangle = 0$. One also notes that

$$(M_b^a)^{\dagger} = M_a^b\,, \qquad (M_{ab})^{\dagger} = M^{ab} \equiv J^{ac} J^{bd} M_{cd}\,, \tag{B.29}$$

$$(N_j^i)^{\dagger} = N_i^j\,, \qquad (T_{ij})^{\dagger} = \pm \overline{T}^{ij}\,, \tag{B.30}$$

using $(J_{ab})^* = J^{ab}$. The Fock space thus decomposes into unitary finite-dimensional tensorial representations of the compact real form of $\mathfrak{l}$, i.e. $\mathfrak{u}(D)$, $\mathfrak{so}(D)$ with $\eta_{ab} = \delta_{ab}$, and $\mathfrak{usp}(D) = \mathfrak{sp}(D;\mathbb{C}) \bigcap \mathfrak{u}(D)$. For the Howe-dual algebra one finds unitary infinite-dimensional representations of the maximally split non-compact real form of $\widetilde{\mathfrak{l}}^+$ and unitary finite-dimensional representations of the compact real form of $\widetilde{\mathfrak{l}}^-$, i.e.

$\mathfrak{l}$	$\widetilde{\mathfrak{l}}^+$	$\widetilde{\mathfrak{l}}^-$	
$\mathfrak{u}(D)$	$\mathfrak{u}(\nu_+)$	$\mathfrak{u}(\nu_-)$	
$\mathfrak{so}(D)$	$\mathfrak{sp}(2\nu_+)$	$\mathfrak{so}(2\nu_-)$	(B.31)
$\mathfrak{usp}(D)$	$\mathfrak{so}(\nu_+,\nu_+)$	$\mathfrak{sp}(2\nu_-)$	

Generalized Fock spaces can be built on anti-vacua that are annihilated by $\bar{\alpha}^{i,a}$ for some values of a. In the case of bosonic oscillators, these modules have a non-degenerate inner product matrix with alternating signature that yields unitary representations of the non-compact real forms of $\mathfrak{l}$. The various Fock-space realizations are subsumed into the Moyal quantization of the oscillator algebra (see, for example, [42]).

$(i + h - 1)$th rows if $\widetilde{w}_i^\pm = \cdots = \widetilde{w}_{i+h-1}^\pm$, and we define the transpose $\Delta^{\rm T}$ of Δ to be the Young diagram with $h_i(\Delta^{\rm T}) = w_i(\Delta)$ (and hence $w_i(\Delta^{\rm T}) = h_i(\Delta)$).

As we shall demonstrate below, the multiplicities

$$\text{mult}_\epsilon^\pm(\Delta|D; \nu_\pm) \in \{0, 1\} \,. \tag{B.35}$$

The Fock space realization is thus completely free of degeneracy in the sense that the correspondence $\lambda \leftrightarrow \widetilde{\lambda}^\pm$ is one-to-one and each dual pair $(\lambda|\widetilde{\lambda}^\pm)$ arises exactly once. Correspondingly, the decomposition of the Schur module reads

$$\mathcal{S}_{D;\nu_\pm}^\pm = \bigoplus_\Delta \text{mult}_\epsilon^\pm(\Delta|D; \nu_\pm) \, (\mathbb{C} \otimes |\Delta\rangle), \qquad |\Delta\rangle = |\lambda(\Delta)|\widetilde{\lambda}^\pm(\Delta)\rangle \,. \tag{B.36}$$

Let us examine more carefully the determination of (B.35).

B.4 Schur modules for $\mathfrak{gl}(D; \mathbb{C})$

In the case of $\mathfrak{l} = \mathfrak{gl}(D; \mathbb{C})$, both $\mathfrak{l}$ and $\widetilde{\mathfrak{l}}^\pm$ leave $\mathcal{F}_{D;\nu_\pm;R}$ invariant, and

$$\mathcal{F}_{D;\nu_\pm;R} = \bigoplus_{\Delta:\ \text{rank}(\Delta)=R} \text{mult}_0^\pm(\Delta|D; \nu_\pm) \, \mathcal{D}^\pm(\lambda(\Delta)|\widetilde{\lambda}^\pm(\Delta)) \,, \tag{B.37}$$

where the highest-weights are given by (B.32)–(B.34) and the multiplicities

$$\text{mult}_0^+(\Delta|D; \nu_+) = \begin{cases} 0 \text{ if } h_1 > \min(\nu_+, D) \,, \\ 1 \text{ else} \end{cases} \tag{B.38}$$

$$\text{mult}_0^-(\Delta|D; \nu_-) = \begin{cases} 0 \text{ if } h_1 > D \text{ or } w_1 > \nu_- \,, \\ 1 \text{ else} \end{cases} \,. \tag{B.39}$$

The vanishing conditions follow immediately from the statistics of the oscillators. To show that the non-vanishing multiplicities are equal to 1, one may use dimension formulae or directly decompose $\mathcal{F}_{D;\nu_\pm;R}^\pm$ under $\mathfrak{gl}(D; \mathbb{C})$.

Calculation of multiplicities using dimension formulae. The total dimension of the right-hand side of (B.37) is given by

$$d_R^\pm(D; \nu_\pm) = \sum_\Delta \text{mult}_0^\pm(\Delta|D; \nu_\pm) \, d^\pm(\Delta|D; \nu_\pm) \,, \tag{B.40}$$

$$d^\pm(\Delta|D; \nu_\pm) = \dim(\mathfrak{gl}(D)|\Delta) \, \dim(\mathfrak{gl}(\nu_\pm)|\widetilde{\Delta}^\pm) \,, \tag{B.41}$$

where the dual Young diagrams

$$\widetilde{\Delta}^+ = \Delta \,, \qquad \widetilde{\Delta}^- = \Delta^{\rm T} \,, \tag{B.42}$$

and

$$\dim(\mathfrak{gl}(N)|\Delta) = \frac{\prod_{(i,j)\in\Delta}(N+i-j)}{|\Delta|} \,, \qquad |\Delta| = \prod_{(i,j)\in\Delta}(w_i+h_j-i-j+1), \tag{B.43}$$

which vanishes in case the height of Δ exceeds N (and it is invariant under insertions and removals of columns of height N although this property is not needed here). Thus $d^\pm(\Delta|D;\nu_\pm)$ vanishes iff $\mathrm{mult}_0^\pm(\Delta|D;\nu_\pm)$ vanishes. Moreover, the denominators on the right-hand side of (B.41) are equal, and

$$d^\pm(\Delta|D;\nu_\pm) = \frac{\prod_{(i,j)\in\Delta}(D-i+j)(\nu_\pm \pm i \mp j)}{|\Delta|^2} = \sum_{m,n=0}^{R} d_{m,n}^\pm(\Delta)D^m(\nu_\pm)^n, \quad (\text{B.44})$$

with $d_{R,R}^\pm(\Delta) = 1/|\Delta|^2$. Thus, from the sum rule

$$\sum_\Delta \frac{1}{|\Delta|^2} = \frac{1}{R!}, \qquad (\text{B.45})$$

which is a consequence of the formula giving the decomposition of the regular representation of the symmetric group S_R in irreps and of the fact that the dimension of the irrep associated with Δ is $R!/|\Delta|$, it follows that the total dimension $d_R^\pm(D;\nu_\pm)$ is a polynomial in D and $\nu_\pm$ with leading behavior given by

$$d_R^\pm(D;\nu_\pm) = \frac{1}{R!}(D^\pm)^R(1+\alpha) + (\text{terms of lower order in } D \text{ and } \nu_\pm), \quad (\text{B.46})$$

for some non-negative integer α. Then, it results that (B.38) and (B.39) must hold in order to reproduce the leading behavior of (B.27), i.e. $\alpha = 0$. We note that the sub-leading coefficients contain generalizations of the sum rule (B.45).

Direct decomposition of $\mathcal{F}_{D;\nu_\pm;R}^\pm$. In the case of bosonic oscillators, the monomial

$$|(m_1)\otimes\cdots\otimes(m_{\nu_+})\rangle = \bar{\alpha}^{1,a_1(m_1)}\cdots\bar{\alpha}^{\nu_+,a_{\nu_+}(m_{\nu_+})}|0\rangle, \qquad \sum_{i=1}^{\nu_+} m_i = R, \quad (\text{B.47})$$

where $\bar{\alpha}^{i,a_i(m_i)} = \bar{\alpha}^{i,a_{i,1}}\cdots\bar{\alpha}^{i,a_{i,m_i}}$, decomposes under $\mathfrak{gl}(D;\mathbb{C})$ into

$$|(m_1)\otimes\cdots\otimes(m_{\nu_+})\rangle = \sum_{\{p_{ij}\}}\prod_{1\leqslant i<j\leqslant\nu_+}(N_i^j)^{p_{ij}}|\Delta\rangle, \qquad (\text{B.48})$$

where:

(i) $|\Delta\rangle$ are carry $\mathfrak{gl}(D;\mathbb{C})$-irreps labelled by admissible Young diagrams Δ;

(ii) $|\Delta\rangle$ are Schur states obeying (B.18) with $\widetilde{\lambda}_i^+$ given by (B.33); and

(iii) $\{p_{ij}\}$ are sets of integers $p_{ij} \in \{0,1,2,\dots\}$ that parameterize the numbers of cells that are lifted from the jth row to the ith row in applying the Littlewood-Richardson rule to $(m_1)\otimes\cdots\otimes(m_{\nu_+})$.

It follows that

$$w_i = m_i + \sum_{i<j} p_{ij} - \sum_{j<i} p_{ji}, \qquad i = 1,\dots,\nu_+, \qquad (\text{B.49})$$

which imply that w_i obey the admissibility conditions

$$w_i \geqslant w_{i+1}, \qquad w_i = 0 \quad \text{for } i > \min(D, \nu_+) . \tag{B.50}$$

The states $\prod_{1 \leqslant i < j \leqslant \nu_+} (N_i^j)^{p_{ij}} |\Delta\rangle$ belong to the Δ-plet of $\mathfrak{gl}(D; \mathbb{C})$ for all admissible $\{p_{ij}\}$, while they are Schur states iff $p_{ij} = 0$. Hence the decomposition (B.48) contains a Schur state iff $m_1 \geqslant m_2 \geqslant m_{\min(D,\nu_+)} \geqslant 0$ and $m_i = 0$ for $i > \min(D, \nu_+)$, in which case its multiplicity is given by 1, which shows (B.38).

Similarly, the case of fermionic oscillators, the monomial

$$|[m_1] \otimes \cdots \otimes [m_{\nu_-}]\rangle = \bar{\alpha}^{1,a_1[m_1]} \cdots \bar{\alpha}^{\nu_-,a_{\nu_-}[m_{\nu_-}]} |0\rangle , \qquad \sum_{i=1}^{\nu_-} m_i = R , \tag{B.51}$$

decomposes under $\mathfrak{gl}(D; \mathbb{C})$ into

$$|[m_1] \otimes \cdots \otimes [m_{\nu_-}]\rangle = \sum_{\{p_{ij}\}} \prod_{1 \leqslant i < j \leqslant \nu_-} (N_i^j)^{p_{ij}} |\Delta\rangle , \tag{B.52}$$

where $|\Delta\rangle$ carry the Δ-plet of $\mathfrak{gl}(D; \mathbb{C})$ and obey (B.18), and

$$h_i = m_i + \sum_{i<j} p_{ij} - \sum_{j<i} p_{ji}, \qquad i = 1, \ldots, \nu_+ , \tag{B.53}$$

subject to the admissibility conditions

$$D \geqslant h_i \geqslant h_{i+1}, \qquad h_i = 0 \quad \text{for } i > \nu_- . \tag{B.54}$$

Hence Schur states arise in (B.52) iff $p_{ij} = 0$, in which case their multiplicity is given by 1, from which (B.39) follows.

B.5 Schur modules for $\mathfrak{so}(D; \mathbb{C})$ and $\mathfrak{sp}(D; \mathbb{C})$

The actions of $\mathfrak{so}(D; \mathbb{C})$ ($\epsilon = +1$) and $\mathfrak{sp}(D; \mathbb{C})$ ($\epsilon = -1$) leave $\mathcal{F}^{\pm}_{D;\nu_\pm;R}$ invariant, while their Howe duals act in representations that in general range over more than one value of R. Correspondingly, for fixed R the $\mathfrak{gl}(D)$-irreps in $\mathcal{F}^{\pm}_{D;\nu_\pm;R}$ decompose into J-traceless states obeying

$$T_{ij} |\Delta\rangle = 0 , \tag{B.55}$$

and J-traces, i.e. states in the image of $\overline{T}^{ij}$. Using the fermionic oscillators, i.e. the anti-symmetric basis of Young projectors, one can show that

$$h_i + h_j + 2 t_{ij} \leqslant D \quad \text{for} \quad \begin{cases} i \neq j & \text{if } \mathfrak{l} = \mathfrak{so}(D; \mathbb{C}) \\ \text{all } i, j & \text{if } \mathfrak{l} = \mathfrak{sp}(D; \mathbb{C}) \end{cases} , \tag{B.56}$$

where t_{ij} denote the total number of traces that have been inserted into columns i and j. The same conditions must hold also in the case of bosonic oscillators. Thus,

$$\mathcal{F}^{\pm}_{D;\nu_\pm} = \bigoplus_{\Delta} \text{mult}^{\pm}_\epsilon(\Delta | D; \nu_\pm) \, \mathcal{D}^{\pm}(\lambda(\Delta) | \widetilde{\lambda}^{\pm}(\Delta)) , \tag{B.57}$$

where the highest weights of $\mathfrak{l}$ and $\widetilde{\mathfrak{l}}^{\pm}$ are given by (B.32)–(B.34) and

$$\mathrm{mult}_\epsilon^\pm(\Delta|D;\nu_-) = \mathrm{mult}_0^\pm(\Delta|D;\nu_-)\,\theta_\epsilon(\Delta)\,, \tag{B.58}$$

with $\theta_\epsilon(\Delta)$ accounting for the condition (B.56) in the case that $t_{ij}=0$, i.e.

$$\theta_\epsilon(\Delta) = \begin{cases} 1 \text{ if (B.56) holds for } t_{ij}=0 \\ 0 \text{ else} \end{cases}\,. \tag{B.59}$$

We note that for $\mathfrak{so}(D;\mathbb{C})$ the highest weight $\widetilde{\lambda}_1^-$ of $\widetilde{\mathfrak{l}}^- = \mathfrak{so}(2\nu_-)$ may become negative, in which case one may redefine the (B.16) by normal-ordering the Howe-dual generators with respect to $\prod_{a=1}^D \bar{\alpha}^{1,a}|0\rangle$ (instead of $|0\rangle$), which leads to an exchange of h_1 by $D - h_1$ and hence $\widetilde{\lambda}_1^-$ by $-\widetilde{\lambda}_1^-$. We also note that if $\nu_- = 2$ then the Schur states of $\mathcal{F}_{D;2;D\nu_-/2}^-$ are annihilated by both T_{12} and $\overline{T}^{12}$ and hence obey $h_1 + h_2 = D$, although they form singlets of $\widetilde{\mathfrak{l}}^-$ only if $h_1 = h_2 = D/2$ and D is even.

C Radial reduction of the background connection

We denote the $\mathfrak{iso}(2, D-1)$-covariant derivative on $\widehat{\mathcal{M}}_{D+1}$ by

$$\widehat{\mathcal{D}} := d - i\left(\widehat{E}^A \widehat{\Pi}_A + \frac{1}{2}\widehat{\Omega}^{AB}\widehat{M}_{AB} \right)\,, \tag{C.1}$$

where $\widehat{\Pi}_A$ are the translation generators, $\widehat{E}^A$ and $\widehat{\Omega}^{AB}$ are the vielbein and $\mathfrak{so}(2, D-1)$-valued connection, respectively. The connection is flat if[14] $\widehat{T}^A := \widehat{\nabla}\widehat{E}^A := d\widehat{E}^A + \widehat{\Omega}^{AB}\widehat{E}_B \approx 0$ and $\widehat{R}^{AB} := d\widehat{\Omega}^{AB} + \widehat{\Omega}^{AC}\widehat{\Omega}_C{}^B \approx 0$. A local foliation of $\widehat{\mathcal{M}}_{D+1}$, as defined in section I.3.7, induces a splitting

$$\widehat{E}^A := \widehat{e}^A + N\widehat{\xi}^A\,, \qquad \widehat{\Omega}^{AB} := \widehat{\omega}^{AB} + N\widehat{\Lambda}^{AB}\,, \tag{C.2}$$

where $\widehat{\xi}^A := i_\xi \widehat{E}^A$ and $\widehat{\Lambda}^{AB} := i_\xi \widehat{\Omega}^{AB}$, which implies $i_\xi \widehat{e}^A = 0$ and $i_\xi \widehat{\omega}^{AB} = 0$. Upon defining

$$\widehat{D} := d - \frac{i}{2}\widehat{\omega}^{AB}\widehat{M}_{AB}\,, \tag{C.3}$$

the flatness conditions decompose into components that are transverse and parallel to i_N as follows

$$(\widehat{D} - N\mathcal{L}_\xi)\widehat{e}^A \approx 0\,, \qquad (\widehat{D} - N\mathcal{L}_\xi)\widehat{\xi}^A - \mathcal{L}_\xi \widehat{e}^A - \widehat{\Lambda}^{AB}\widehat{e}_B \approx 0\,, \tag{C.4}$$

$$d\widehat{\omega}^{AB} + \widehat{\omega}^{AC}\widehat{\omega}_C{}^B - N\mathcal{L}_\xi \widehat{\omega}^{AB} \approx 0\,, \qquad (\widehat{D} - N\mathcal{L}_\xi)\widehat{\Lambda}^{AB} - \mathcal{L}_\xi \widehat{\omega}^{AB} \approx 0\,. \tag{C.5}$$

---

[14]Although not used here, we note that the flat vielbein can be expressed locally as $\widehat{E}^A = \widehat{\nabla}\widehat{V}^A$. In foliations with maximally symmetric leaves and constant $\widehat{\xi}^A$, the gauge function can be chosen to be $\widehat{V}^A = \lambda^{-1}\widehat{\xi}^A$.

There remains a manifest covariance under $O(2, D-1)$ gauge transformations with parameters annihilated by $\mathcal{L}_\xi$. We denote this gauge group by $O(2, D-1)_{\text{leaf}}$. Maximally symmetric leaves arise from foliations obeying

$$\widehat{\Lambda}^{AB} \stackrel{!}{=} 0, \quad \mathcal{L}_\xi \widehat{e}^A \stackrel{!}{=} \lambda(L) \widehat{e}^A, \quad \mathcal{L}_\xi \widehat{\xi}^A \stackrel{!}{=} 0, \tag{C.6}$$

which implies $\widehat{D} \equiv \widehat{\nabla}$ and the local relations

$$\widehat{D}\widehat{e}^A \approx \lambda N \widehat{e}^A, \quad \widehat{\xi}_A \widehat{e}^A \approx 0, \quad \widehat{D}\widehat{\xi}^A \approx \lambda \widehat{e}^A, \quad d\lambda + \lambda^2 N \approx 0, \tag{C.7}$$

$$d\widehat{\omega}^{AB} + \widehat{\omega}^{AC}\widehat{\omega}_C{}^B \approx 0, \quad \mathcal{L}_\xi \widehat{\omega}^{AB} \approx 0, \quad (\mathcal{L}_\xi)^2 \widehat{e}^A \approx 0. \tag{C.8}$$

One may choose

$$\lambda = L^{-1}, \tag{C.9}$$

and use local $O(2, D-1)_{\text{leaf}}$-symmetry to bring $\widehat{\xi}^A$ to a locally constant vector, i.e.

$$d\widehat{\xi}^A \stackrel{!}{=} 0 \qquad \text{(gauge-fix } O(2, D-1)_{\text{leaf}}\text{)} \tag{C.10}$$

whose residual local symmetry group we denote by $G_{\text{leaf}}(\widehat{\xi}^2)$. The global decomposition is

$$\widehat{\mathcal{M}}_{D+1} = \widehat{\mathcal{M}}_{D+1}^{(-1)} \cup \widehat{\mathcal{M}}_D^{(0)} \cup \widehat{\mathcal{M}}_{D+1}^{(1)}, \tag{C.11}$$

where $\widehat{\mathcal{M}}_{D(k)}^{(k)}$ are regions of dimension $D(k)$ foliated with maximally symmetric leaves with $\widehat{\xi}^2 = k$ and local $G_{\text{leaf}}(k)$ symmetry. In $\widehat{\mathcal{M}}_{D+1}^{(-1)}$ the projector $\widehat{\xi}_A \mathbb{P}_B^A := 0$, $\mathbb{P}_B^A \widehat{\xi}^B := 0$ obeys $\mathbb{P}_B^A := (0, \mathbb{P}_B^a)$ where the index a transforms as a vector under residual local $G_{\text{leaf}}(-1) \cong O(1, D-1)_{\text{leaf}}$ transformations. Defining

$$\omega^{AB} := \widehat{\omega}^{AB} + \lambda(\widehat{e}^A \widehat{\xi}^B - \widehat{\xi}^A \widehat{e}^B), \tag{C.12}$$

then the local relations imply that $(k = \widehat{\xi}^2 = -1)$

$$\widehat{\xi}_A \omega^{AB} \approx 0, \quad d\omega^{AB} + \omega^{AC}\omega_C{}^B + \lambda^2 \widehat{e}^A \widehat{e}^B \approx 0, \tag{C.13}$$

and one identifies the leaves as AdS_D spacetimes of radius L with canonical flat $\mathfrak{so}(2, D-1)$-valued connections

$$e^a := i_L^* \, \mathbb{P}_A^a \, \widehat{e}^A, \quad \omega^{ab} := i_L^* \, \mathbb{P}_A^a \, \mathbb{P}_B^b \, \omega^{AB}, \tag{C.14}$$

as defined in (A.4). Skvortsov's master-field equations contain the $\mathfrak{iso}(2, D-1)$-covariant derivatives $(i = 1, \ldots, \nu)$

$$\widehat{\mathcal{D}}_{(i)} := \widehat{\nabla} - i\widehat{E}_{(i)} = d - \frac{i}{2}\widehat{\Omega}^{AB}\widehat{M}_{AB} - i\widehat{E}^A \widehat{\beta}_{A,(i)} \tag{C.15}$$

$$= d - \frac{i}{2}\left(\omega^{AB} + 2\lambda\widehat{\xi}^A\widehat{e}^B\right)\widehat{M}_{AB} - i\widehat{e}^A\widehat{\beta}_{A,(i)} - iN\widehat{\xi}_{(i)}, \tag{C.16}$$

where $\widehat{\xi}_{(i)} := \widehat{\xi}^A \widehat{\beta}_{A,(i)}$. Radial reduction can be analyzed directly on $\widehat{\mathcal{M}}_{D+1}$ using

$$\widehat{\mathcal{D}}_{(i)} := \widehat{\nabla} - i\widehat{e}_{(i)} - iN\widehat{\xi}_{(i)}, \quad e_{(i)} := \widehat{e}^A \widehat{\beta}_{A,(i)}, \tag{C.17}$$

whilst the harmonic expansion and flat limit can be analyzed on $AdS_D(L)$ using

$$\mathcal{D}_{(i)} := i_L^* \widehat{\mathcal{D}}_{(i)} := \nabla - ie^a P_{a,(i)}, \quad P_{a,(i)} := \lambda \widehat{\xi}^B \widehat{M}_{Ba} + \widehat{\beta}_{a,(i)}, \tag{C.18}$$

where $[\widehat{\beta}_{A,(i)}, \widehat{\beta}_{B,(i)}] = 0$ and $[P_{a,(i)}, P_{b,(i)}] = i\lambda^2 \widehat{M}_{ab}$.

D Tensorial content of the σ^--chains with $h_1 = 1$ and $q + g = 2, 3$

The $\mathfrak{m}$-content of the σ^--chain in the case of $h_1 = 1$, $s_1 - s_2 \geqslant 4$ is given for $q + g = 2$ by

$$
X^{\mathbf{1}}(R_{]a(2)}) \in \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(3) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(2, 1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}(2) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \, \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(1) \end{bmatrix} , \tag{D.1}
$$

$$
R^{[2]}(R_{\alpha(1)}) \in \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(2, 1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}[3] \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \, \widetilde{\otimes}(1) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 \\ \Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \, \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \, \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \, \widetilde{\otimes}(1) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \, \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \end{bmatrix} , \tag{D.2}
$$

$$
Z^{[3]}(R_{\alpha(0)}) \in \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}[3] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(1) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \, \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[2]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \, \widetilde{\otimes}(1) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \, \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \, \widetilde{\otimes}(1) \end{bmatrix} , \tag{D.3}
$$

and for $q + g = 3$ by

$$
X^{\mathbf{1}}(R_{\alpha(3)}) \in \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}(3) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(4) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(3, 1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}(3) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \, \widetilde{\otimes}(3) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(2) \end{bmatrix} , \tag{D.4}
$$

$$
R^{[2]}(R_{\alpha(2)}) \in \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}(3) \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}(2, 1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(3, 1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}[3, 1] \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}(3) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}[2, 1] \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \, \widetilde{\otimes}(2) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \, \widetilde{\otimes}(3) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \, \widetilde{\otimes}[2, 1] \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ \Xi \, \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}(2) \end{bmatrix}
$$
$$
\oplus \begin{bmatrix} s_1 - 1 \\ \Xi \, \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \, \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \, \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \, \widetilde{\otimes}(2) \end{bmatrix}
$$

$$\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix}, \tag{D.5}$$

$$Z^{[3]}(R_{\alpha(1)}) \in \begin{bmatrix} s_1 \\ \Xi \ \widetilde{\otimes}(2,1) \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ \Xi \ \widetilde{\otimes}[3] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \ \widetilde{\otimes}[3,1] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \ \widetilde{\otimes}[4] \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 - 1 \\ \Xi \ \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \ \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \ \widetilde{\otimes}[2,1] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \ \widetilde{\otimes}[3] \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \ \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \ \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \ \widetilde{\otimes}[2,1] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \ \widetilde{\otimes}[3] \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \ \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \ \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 - 2 \\ \Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ \Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \ \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 \\ i_{[2]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \ \widetilde{\otimes}(2) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \ \widetilde{\otimes}[2] \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 - 2 \\ i_{[2]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[3]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \end{bmatrix}, \tag{D.6}$$

$$Z_3^{[4]}(R_{\alpha(0)}) \in \begin{bmatrix} s_1 \\ \Xi \ \widetilde{\otimes}[3] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \ \widetilde{\otimes}[4] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ \Xi \ \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ \Xi \ \widetilde{\otimes}[3] \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 \\ i_{[1]}\Xi \ \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \ \widetilde{\otimes}[3] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[1]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[1]}\Xi \ \widetilde{\otimes}[2] \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 \\ i_{[2]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \ \widetilde{\otimes}[2] \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[2]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 2 \\ i_{[2]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix}$$

$$\oplus \begin{bmatrix} s_1 \\ i_{[3]}\Xi \end{bmatrix} \oplus \begin{bmatrix} s_1 - 1 \\ i_{[3]}\Xi \ \widetilde{\otimes}(1) \end{bmatrix}. \tag{D.7}$$

E Mixed-symmetry gauge fields and singleton composites

The bosonic singletons $\mathfrak{D}_s \equiv D(\epsilon_0 + s; ([s;h]))$ consist of states $|e_n; ([s+n;1], [s;h-1])\rangle$, $n = 0,1,\ldots,$ of energy $e_n = \epsilon_0 + s + n$ and $\mathfrak{so}(D-1)$ spin $([s+n;1], [s;h-1])$ where $\epsilon_0 = h-1 = (D-3)/2$ and $s > 0$ requires D to be odd. The tensor product $\mathfrak{D}_{s_1} \otimes \cdots \otimes \mathfrak{D}_{s_P}$ consists of states with energy

$$e = \sum_{i=1}^{P}(\epsilon_0 + s_i + n_i), \tag{E.1}$$

and spin

$$(s_1 + n_1, s_1, \ldots, s_1) \otimes \cdots \otimes (s_P + n_P, s_P, \ldots, s_P) = \bigoplus_{t_1,\ldots,t_\nu}(t_1,\ldots,t_\nu), \tag{E.2}$$

where $t_1 \leqslant \sum_{i=1}^{P}(s_i + n_i)$. Thus

$$e \geqslant P\epsilon_0 + t_1 \ . \tag{E.3}$$

The ground states of unitary[15] massless representations have

$$e = t_1 + D - 2 - h_1 \,, \tag{E.4}$$

where h_1 is the height of the first block of $(t_1, \ldots, t_\nu)$, i.e. $t_1 = \cdots = t_{h_1} > t_{h_1+1}$. Such states fit inside P-fold product only if $P\epsilon_0 + t_1 \leqslant t_1 + D - 2 - h_1$, that is $P \leqslant 2(D - 2 - h_1)/(D - 3)$. Since $h_1 \geqslant 1$ and $P \geqslant 2$ it follows that

$$h_1 = 1 \,, \qquad P = 2 \,, \tag{E.5}$$

that is, only unitary mixed-symmetry massless fields with $h_1 = 1$ and with at most 6 blocks can be singleton composites. Since $h_1 = 1$ the corresponding ASV potentials are 1-forms, for which there could be a standard non-abelian closure of the gauge algebra.

References

[1] N. Boulanger, C. Iazeolla and P. Sundell, *Unfolding mixed-symmetry fields in AdS and the BMV conjecture: I. General formalism*, *JHEP* **07** (2009) 013 [arXiv:0812.3615] [SPIRES].

[2] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *On the frame-like formulation of mixed-symmetry massless fields in (A)dS(d)*, *Nucl. Phys.* **B 692** (2004) 363 [hep-th/0311164] [SPIRES].

[3] E.D. Skvortsov, *Mixed-symmetry massless fields in Minkowski space unfolded*, *JHEP* **07** (2008) 004 [arXiv:0801.2268] [SPIRES].

[4] R.R. Metsaev, *Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time*, *Phys. Lett.* **B 354** (1995) 78 [SPIRES].

[5] R.R. Metsaev, *Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space*, hep-th/9810231 [SPIRES].

[6] A. Fotopoulos and M. Tsulaia, *Gauge invariant Lagrangians for free and interacting higher spin fields. A review of the BRST formulation*, *Int. J. Mod. Phys.* **A 24** (2009) 1 [arXiv:0805.1346] [SPIRES].

[7] Y.M. Zinoviev, *Toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields*, *Nucl. Phys.* **B 812** (2009) 46 [arXiv:0809.3287] [SPIRES].

[8] I.L. Buchbinder, V.A. Krykhtin and L.L. Ryskina, *BRST approach to Lagrangian formulation of bosonic totally antisymmeric tensor fields in curved space*, *Mod. Phys. Lett.* **A 24** (2009) 401 [arXiv:0810.3467] [SPIRES].

[9] A.A. Reshetnyak, *Nonlinear operator superalgebras and BFV-BRST operators for Lagrangian description of mixed-symmetry HS fields in AdS spaces*, arXiv:0812.2329 [SPIRES].

[10] M.A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, *Phys. Lett.* **B 209** (1988) 491 [SPIRES].

---

[15]The tensor product of singletons is unitary and hence cannot contain the non-unitary massless representations with $e < t_1 + D - 2 - h_1$.

[11] M.A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Annals Phys.* **190** (1989) 59 [SPIRES].

[12] M.A. Vasiliev, *Unfolded representation for relativistic equations in* $(2 + 1)$ *anti-de Sitter space*, *Class. Quant. Grav.* **11** (1994) 649 [SPIRES].

[13] D. Sullivan, *Infinitesimal computations in topology*, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977) 269.

[14] R. D'Auria and P. Fré, *Geometric supergravity in* $D = 11$ *and its hidden supergroup*, *Nucl. Phys.* **B 201** (1982) 101 [*Erratum ibid.* **B 206** (1982) 496] [SPIRES].

[15] R. D'Auria, P. Fré, P.K. Townsend and P. van Nieuwenhuizen, *Invariance of actions, rheonomy and the new minimal* $N = 1$ *supergravity in the group manifold approach*, *Ann. Phys.* **155** (1984) 423 [SPIRES].

[16] P. van Nieuwenhuizen, *Free graded differential superalgebras*, invited talk given at 11th *International Colloq. on Group Theoretical Methods in Physics, Istanbul Turkey August* 23–28 1982 [SPIRES].

[17] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in* $(3 + 1)$*-dimensions*, *Phys. Lett.* **B 243** (1990) 378 [SPIRES].

[18] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys. Lett.* **B 567** (2003) 139 [hep-th/0304049] [SPIRES].

[19] L. Brink, R.R. Metsaev and M.A. Vasiliev, *How massless are massless fields in AdS(d)*, *Nucl. Phys.* **B 586** (2000) 183 [hep-th/0005136] [SPIRES].

[20] J.M.F. Labastida, *Massless particles in arbitrary representations of the Lorentz group*, *Nucl. Phys.* **B 322** (1989) 185 [SPIRES].

[21] X. Bekaert and N. Boulanger, *Mixed symmetry gauge fields in a flat background*, published in the *Proceedings of the International Conference on Supersymmetries and Quantum Symmetries SQS* 03, *Dubna Russia July* 24–29 2003 [hep-th/0310209] [SPIRES].

[22] X. Bekaert and N. Boulanger, *Tensor gauge fields in arbitrary representations of GL(D,R). II: quadratic actions*, *Commun. Math. Phys.* **271** (2007) 723 [hep-th/0606198] [SPIRES].

[23] X. Bekaert and N. Boulanger, *Tensor gauge fields in arbitrary representations of GL(D,R): duality and Poincaré lemma*, *Commun. Math. Phys.* **245** (2004) 27 [hep-th/0208058] [SPIRES].

[24] X. Bekaert and N. Boulanger, *On geometric equations and duality for free higher spins*, *Phys. Lett.* **B 561** (2003) 183 [hep-th/0301243] [SPIRES].

[25] V. Bargmann and E.P. Wigner, *Group theoretical discussion of relativistic wave equations*, *Proc. Nat. Acad. Sci.* **34** (1948) 211 [SPIRES].

[26] A. Campoleoni, D. Francia, J. Mourad and A. Sagnotti, *Unconstrained Higher spins of mixed symmetry. I. Bose fields*, *Nucl. Phys.* **B 815** (2009) 289 [arXiv:0810.4350] [SPIRES].

[27] E.D. Skvortsov, *Frame-like actions for massless mixed-symmetry fields in Minkowski space*, *Nucl. Phys.* **B 808** (2009) 569 [arXiv:0807.0903] [SPIRES].

[28] M.A. Vasiliev, *'Gauge' form of description of massless fields with arbitrary spin* (in Russian), *Yad. Fiz.* **32** (1980) 855 [*Sov. J. Nucl. Phys.* **32** (1980) 439] [SPIRES].

[29] C. Fronsdal, *Massless fields with integer spin*, *Phys. Rev.* **D 18** (1978) 3624 [SPIRES].

[30] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *Lagrangian formulation for free mixed-symmetry bosonic gauge fields in (A)dS(d)*, *JHEP* **08** (2005) 069 [`hep-th/0501108`] [SPIRES].

[31] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *Frame-like formulation for free mixed-symmetry bosonic massless higher-spin fields in AdS(d)*, `hep-th/0601225` [SPIRES].

[32] K.B. Alkalaev, *Two-column higher spin massless fields in AdS(d)*, *Theor. Math. Phys.* **140** (2004) 1253 [*Teor. Mat. Fiz.* **140** (2004) 424] [`hep-th/0311212`] [SPIRES].

[33] O.V. Shaynkman and M.A. Vasiliev, *Scalar field in any dimension from the higher spin gauge theory perspective*, *Theor. Math. Phys.* **123** (2000) 683 [`hep-th/0003123`] [SPIRES].

[34] M.A. Vasiliev, *Algebraic aspects of the higher spin problem*, *Phys. Lett.* **B 257** (1991) 111 [SPIRES].

[35] M.A. Vasiliev, *Higher spin gauge theories: star-product and AdS space*, `hep-th/9910096` [SPIRES].

[36] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, `hep-th/0503128` [SPIRES].

[37] P.J. Olver, *Differential hyperforms I*, University of Minnesota report 82-101, Minneapolis U.S.A. (1982).

[38] P.J. Olver, *Invariant theory and differential equations*, in *Invariant theory*, S. Koh ed., Springer-Verlag, Germany (1987), pg. 62.

[39] M. Dubois-Violette and M. Henneaux, *Generalized cohomology for irreducible tensor fields of mixed Young symmetry type*, *Lett. Math. Phys.* **49** (1999) 245 [`math.QA/9907135`] [SPIRES].

[40] M. Dubois-Violette and M. Henneaux, *Tensor fields of mixed Young symmetry type and N-complexes*, *Commun. Math. Phys.* **226** (2002) 393 [`math.QA/0110088`] [SPIRES].

[41] S. Deser and A. Waldron, *Partial masslessness of higher spins in (A)dS*, *Nucl. Phys.* **B 607** (2001) 577 [`hep-th/0103198`] [SPIRES].

[42] C. Iazeolla and P. Sundell, *A fiber approach to harmonic analysis of unfolded higher-spin field equations*, *JHEP* **10** (2008) 022 [`arXiv:0806.1942`] [SPIRES].

[43] S. Deser and R.I. Nepomechie, *Gauge invariance versus masslessness in de Sitter space*, *Ann. Phys.* **154** (1984) 396 [SPIRES].

[44] Y.M. Zinoviev, *On massive high spin particles in (A)dS*, `hep-th/0108192` [SPIRES].

[45] E.D. Skvortsov and M.A. Vasiliev, *Geometric formulation for partially massless fields*, *Nucl. Phys.* **B 756** (2006) 117 [`hep-th/0601095`] [SPIRES].

[46] M.A. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **12** (2004) 046 [`hep-th/0404124`] [SPIRES].

[47] K.B. Alkalaev, M. Grigoriev and I.Y. Tipunin, *Massless Poincaré modules and gauge invariant equations*, `arXiv:0811.3999` [SPIRES].